Probability and Counting Techniques

a Teacher Training Session held at Michaelhouse

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The **probability** of an event is a measure of the likelihood of it happening. We express this in symbols as $P(E)$.

$0 \leq P(E) \leq 1$

The **sample space** is the set of all possible outcomes.

Probabilities are often expressed as **percentages** – the SI unit of fractions!

The probability of an event happening is calculated by dividing the number of ways the event can happen by the total number of possible outcomes.

$P(E) = \dfrac{n(E)}{n(S)}$	This formula **IS** provided

The probability of the event not happening is denoted by $P(\overline{E})$. $\overline{E}$ or E' is called the **complement** of E.

$P(E) + P(\overline{E}) = 1$	This formula is **NOT** provided

The complement of "**at least one**" is none!

It two events are **independent** it means that the happening of one does not affect the probability of the other happening.

A and B independent $\Leftrightarrow$ $P(A \cap B) = P(A \text{ and } B) = P(A) \times P(B)$	This formula is **NOT** provided

We can use this to test for independence.

Mutually exclusive events **<u>cannot</u>** both happen.

For example, a learner being in Grade 10 and in Grade 12.

Non-mutually exclusive events **<u>can</u>** both happen.

For example, a learner being female and in Grade 12.

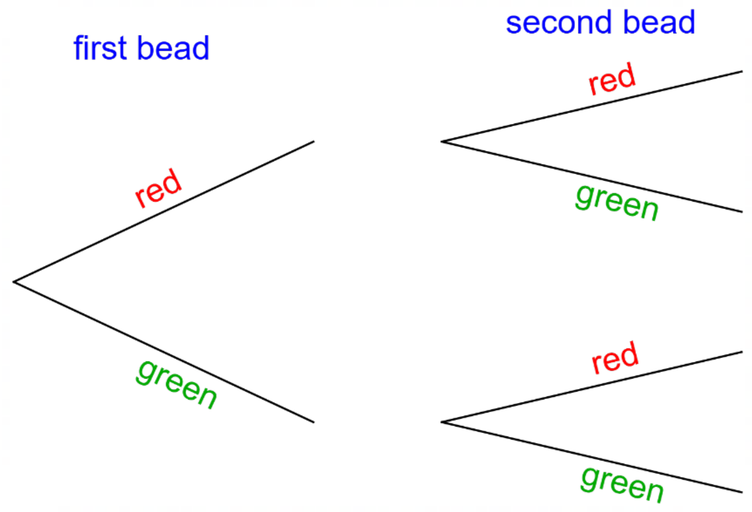
A and B are mutually exclusive $\Leftrightarrow$ $P(A \cap B) = P(A \text{ and } B) = 0$	This formula is **NOT** provided

The probability of one thing **OR** another happening. Remember **OR** includes **BOTH**!

$P(A \cup B) = P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$	This formula **IS** provided

Probability Trees are useful for working out the probabilities of compound events (a combination of more than 1 event).

For example, suppose a bag contains three red beads and 1 green bead. We draw out two beads without replacement.



Whenever a branch splits the probabilities of all the choices must add to 1!

If we want the probability of the combination of events at the end of a branch, we simply multiply the probabilities en route!

To calculate composite probabilities, we simply add the probabilities of the various ways in which they can occur.

e.g. P(one of each colour) =

Venn Diagrams are another way in which probabilities can be written onto a diagram.

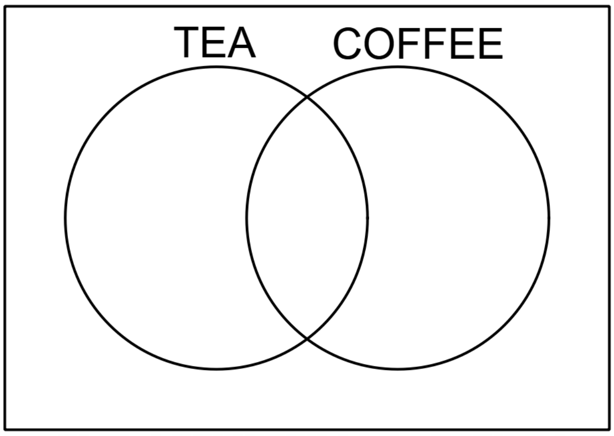
When asked to draw a venn diagram we typically start in the middle (the intersection) and work outwards.

For example, suppose the following probabilities exist:

$P(\text{liking tea}) = 40\%$

$P(\text{liking coffee}) = 65\%$

$P(\text{liking tea and coffee}) = 30\%$



Are the events "liking tea" and "liking coffee":

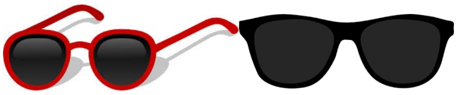
Independent?

Mutually exclusive?

Counting Techniques

Learners (and teachers) often find this difficult so it is best introduced with concrete examples:

If I have 2 pairs of sunglasses (red and black) and 3 items of headwear (blue, red and brown)



How many different combinations can I wear?

Let's write them down in the table:

Glasses	Headwear

no. of choices for glasses × no. of choices for headwear

$\square \times \square =$

What if I also had a choice of two scarves? How many combinations would I then have?

The **FUNDAMENTAL COUNTING PRINCIPLE** says that we multiply numbers of choices!

What if I decide I do not need to wear sunglasses or an item of headwear?

Glasses	Headwear

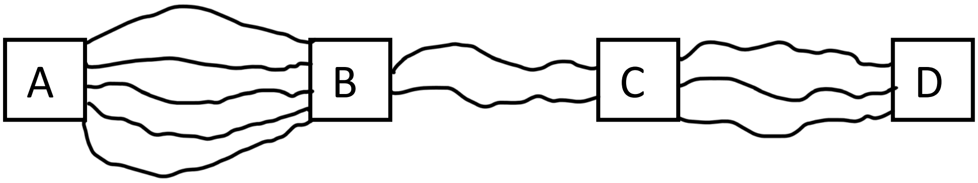
no. of choices for glasses × no. of choices for headwear

$\square \times \square =$

Now try these questions.

Scenario 2

If there are 5 roads between town A and town B and 2 roads between town B and town C and 3 roads between town C and town D then how many different ways can we get from town A to town D?



no. of routes for first leg
×
no. of routes for second leg
×
no. of routes for third leg

$= \square \times \square \times \square$

$=$

Scenario 3

A restaurant offers 3 starters, 4 main courses and 2 puddings.

STARTERS	**MAINS**	**PUDDINGS**
		

How many different meals are possible?

What if we only have 2 of the three courses (in the normal order)?

If we don't have a starter we have $\square \times \square =$ options

If we don't have a starter we have $\square \times \square =$ options

If we don't have a starter we have $\square \times \square =$ options

TOTAL of $\square$ two-course meals

Scenario 4

Consider the following crossword puzzle. How many ways can it be completed?

1	■	■
A	■	■
2 T		N

1. A _____ can be considered a nuisance
2. A _____ is considered a good score in a certain sport

For clue number 1 we can choose __________ or __________ or __________

For clue number 2 we can choose __________ or __________

So there are a total of

$\square \times \square =$

of completing the crossword.

Scenario 5

How many factors does 24 have?

$24 = 2^3 \times 3$

Clearly 5 is not a factor of 24

Clearly $3^2 = 9$ is not a factor of 24

All factors of 24 are made up by choosing combinations of some 2s (a maximum of 3) and some 3s (a maximum of 1)

Remember, we can choose none!

Number of choices for $2s$ $\times$ number of choices for $3s$

$= \square \times \square$

$=$

So, 24 has _________ factors.

They are written upside-down at the bottom of the page so you can check that this method works!

factors of 24 = {1 ; 2 ; 3 ; 4 ; 6 ; 8 ; 12 ; 24}

Scenario 6

The new number plates in KZN are of the form:

Two letters followed by two digits followed by two letters.

Vowels are excluded but digits and letters can be repeated.

How many different licence plates are possible?

Consider putting numbers of choices in the boxes:

$\square \times \square \times \square \times \square \times \square \times \square =$

Scenario 7

From your envelope take the following:

The blue rectangle
The orange rectangle
The yellow rectangle

In how many ways can they be arranged?

Write them down in the spaces below.

You can use the initial letters as a shortcut – e.g. BOY means Blue Orange Yellow

How many choices did you have for the first position? ____________

How many choices did you have for the second position? ____________

How many choices did you have for the third position? ____________

Scenario 8

Four children run a race. Assuming there are no ties, in how many different orders can they finish?

Position	1^{st}	2^{nd}	3^{rd}	4^{th}
No. of choices				

There are a total of

$\square \times \square \times \square \times \square =$

different combinations in which they can finish.

Scenario 9

In how many ways can I arrange 5 books on my shelf?

Position	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}
No. of choices					

There are a total of

$\square \times \square \times \square \times \square \times \square =$

different combinations in which the books can be arranged.

The number of ways we can arrange *n* **distinct (different)** objects is

What if we wish to **keep some objects next to one another**

From your envelope take:

The blue rectangle
The orange rectangle
The yellow rectangle
A green rectangle
The green circle
The green triangle
The frame

Suppose we now want to count the number of arrangements of the above items but **with the green items together.**

Put the green items **inside the frame** and arrange the others outside

If we consider the items in the frame as 1 object then we have _____ objects which can be

arranged in __________ ways

BUT, for each of these, the objects inside the frame can be arranged in _______ ways.

So, we have a total of

$$\square \times \square =$$

Let's look at another example of this.

On my book shelf I have 16 books. Six are by John Grisham and five are by Wilbur Smith.

In how many ways can I arrange them altogether?

In how many ways can I arrange them if I wish to keep books by the same author together?

Consider putting a rubber band round the books we wish to keep together

A possible arrangement is as follows:

$$b_3 \boxed{G_1\ G_4\ G_3\ G_5\ G_2\ G_6}\ b_1\ b_5\ b_2 \boxed{S_2\ S_4\ S_1\ S_3\ S_5}\ b_4$$

If we consider the books in the band as one "book"

Then we have a total of ___________ "books"

Which can be arranged in ____________ ways

BUT

For each of these we can arrange the John Grisham books in __________ ways

and the Wilbur Smith books in ________ ways

So, we have a total of : _____________________________________ arrangements.

What if we wish to arrange **objects some of which are identical?**

From your envelope take:

The blue rectangle
The orange rectangle
The yellow rectangle
The three green rectangles

How many different arrangements can you make given that the three green rectangles are different (each has a small number on it)?

____

Generate one possible arrangement:

For example:

$\boxed{G_2}\ \boxed{B}\ \boxed{Y}\ \boxed{G_1}\ \boxed{O}\ \boxed{G_3}$

Now, if we consider all the green cards as being identical (by turning them over) then our answer will be many times too big.

How many times too big?

Turn them back over in order that you can see the numbers

How many different ways can we arrange the 3 greens in the pattern above? We are leaving the Blue, Orange and Yellow in their original positions.

____

Here they are:

$\boxed{G_2}\ \boxed{B}\ \boxed{Y}\ \boxed{G_1}\ \boxed{O}\ \boxed{G_3}$	$\boxed{G_2}\ \boxed{B}\ \boxed{Y}\ \boxed{G_3}\ \boxed{O}\ \boxed{G_1}$	$\boxed{G_3}\ \boxed{B}\ \boxed{Y}\ \boxed{G_1}\ \boxed{O}\ \boxed{G_2}$
$\boxed{G_3}\ \boxed{B}\ \boxed{Y}\ \boxed{G_2}\ \boxed{O}\ \boxed{G_1}$	$\boxed{G_1}\ \boxed{B}\ \boxed{Y}\ \boxed{G_2}\ \boxed{O}\ \boxed{G_3}$	$\boxed{G_1}\ \boxed{B}\ \boxed{Y}\ \boxed{G_3}\ \boxed{O}\ \boxed{G_2}$

So, our original answer is ____________ times too big and the number of ways we

can arrange 6 objects if 3 of them are identical is:

____

Let's consider two more examples of this….

In how many ways can I arrange 3 green beads, 4 pink beads, a yellow bead, a black bead and a blue bead?

How many different "words" can we make using all the letters of the word

senseless

If we have n objects of which a are identical to one another and b are identical to one another and c are identical to one another then we can arrange them in $\dfrac{n!}{a!b!c!}$ different ways.

Time to practise!

1. A die is thrown and a card is drawn from a pack. Determine the following probabilities:

   a. $P(3$ on die $\cap$ a red card) (2)

   b. P(prime on die ∩ a heart) (2)

   c. P(factor of 9 on die ∪ a black card) (3)

   d. P(even on die ∪ a spade) (3)

2. Two friends are comparing their chances of getting a ticket to a certain concert. Suppose Thandi has a 40% chance, Bongi has a 16% chance and there is 6,4% chance that both get a ticket.

   Are their chances of getting a ticket:

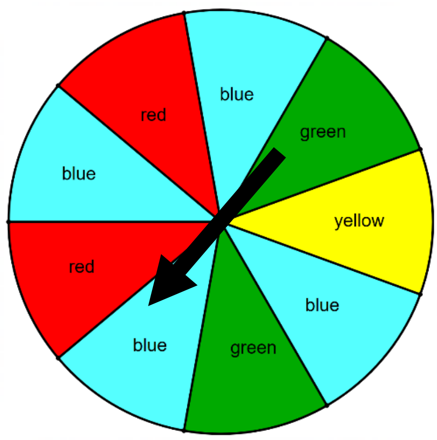
   a. mutually exclusive? Justify. (2)

   b. Independent? Justify. (2)

3. Suppose event A is six times more likely to happen than event B and that they are Mutually exclusive. If $P(A \cup B) = 0{,}84$ then determine $P(B)$ (4)

4. I spin the spinner below. It has nine equally-sized sectors.

Determine:



a. P(red')

b. P(green ∪ blue)

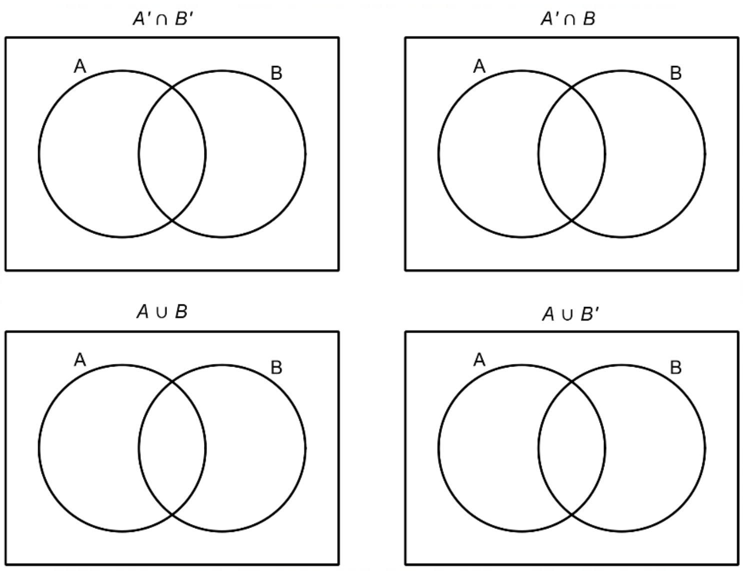
c. $P(2$ greens in 2 spins$)$

d. $P(1$ red in 2 spins$)$

e. P(at least one blue in 5 spins)

f. The minimum number of times I must spin it to be at least 90% sure of getting at least one yellow.

5. Shade the requested areas on the venn diagrams below: (8)

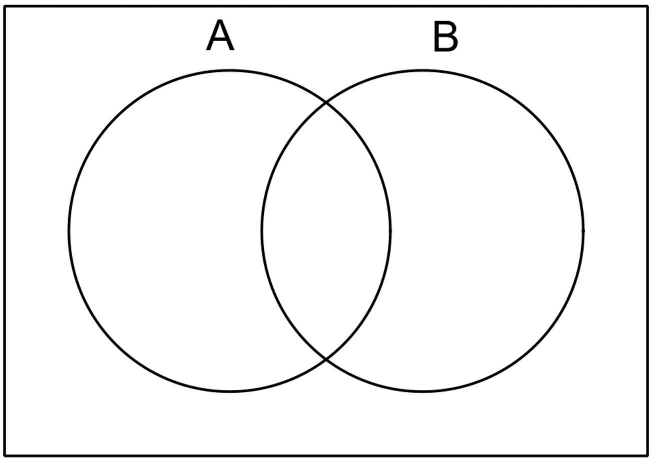


6. The probability that Phumlani finishes the Comrades is 0,8 while the probability that Sanele finishes is 0,7. If the probability that either Phumlani or Sanele finishes is 94% then determine the probability that both finish and determine whether or not their chances of finishing are independent or not. (4)

7. Given:

$P(A' \cap B) = 0{,}6$ $\quad$ $P(A \cap B) = 0{,}2$ $\quad$ $P(A) = 0{,}3$

Draw a venn diagram. (3)



Now use it to determine:

a. If A and B are mutually exclusive? Justify your answer. (1)

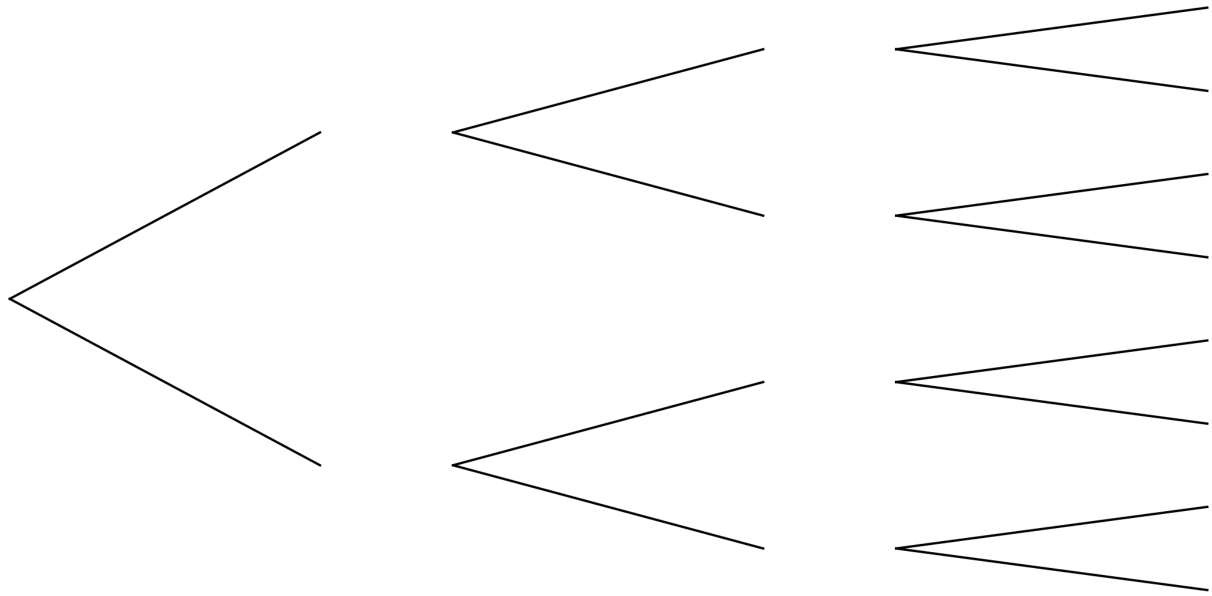
b. If A and B are independent? Justify your answer. (2)

c. $P(A' \cap B')$ (1)

d) $P(A \cup B)$ (1)

8. A music play list on my phone has 9 songs. It plays them in random order and will not replay a song until all the songs have played. Four of the songs are by Taylor Swift.

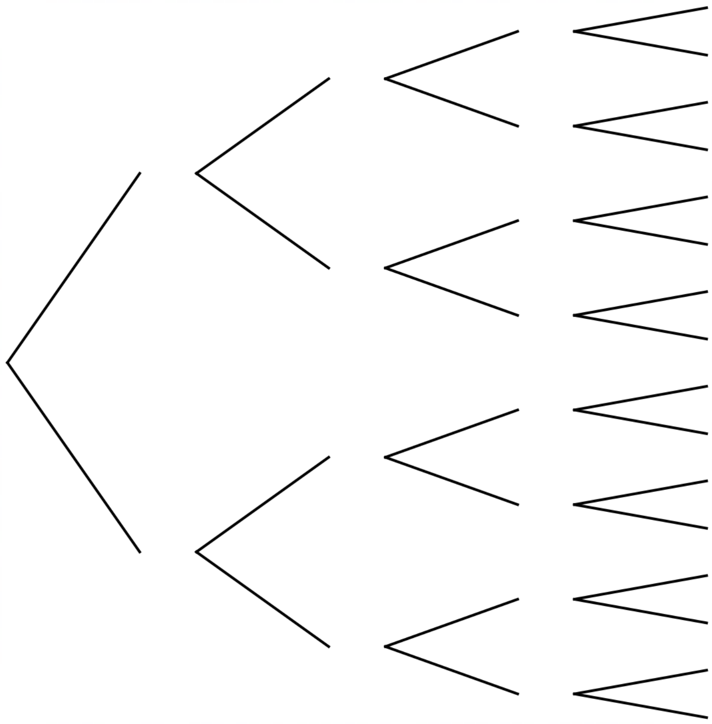
Draw a tree diagram and use it to determine the probability that exactly 2 of the next three songs are by Taylor Swift. (5)



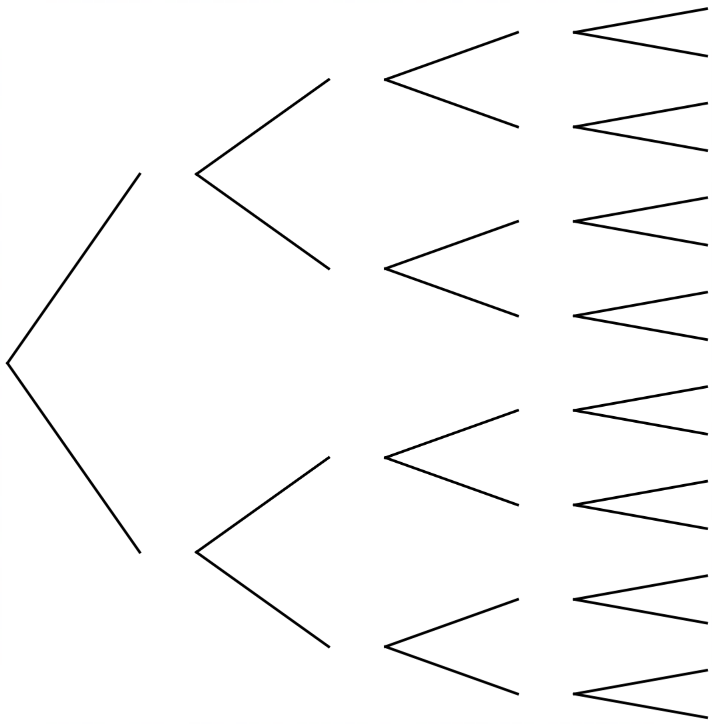
8. A bag contains 1 green bead and 3 red beads. John and Thandi take turns **removing** A bead from the bag with Thandi going first. The person who draws the green bead wins.

Determine the probability that Thandi wins.

Hint: use a tree diagram.



9. Repeat question 8 but this time with the bead being **replaced** after each turn. (6)



10. A computer code takes the form of 3 digits which may not be repeated followed by 2 letters which may be repeated.

An example of such a code is 015AA

a. How many such codes are possible? (3)

b. Suppose that twenty billion (20 000 000 000) unique codes are needed. It is decided to add extra letters. **Calculate** the **minimum** number of extra letters which will be needed. (5)

11. Adam is forming 5 digit numbers by choosing from the digits

2 ; 3 ; 4 ; 6 ; 7 ; 8 and 9

There is only one of each of the above digits.

a. How many numbers can Adam form? (2)

b. If one of the numbers formed by Adam is chosen at random what is the probability that it is even AND exceeds 70 000? (5)

12. Fundi arranges her eight books randomly on her shelf. What is the probability that her three Maths books are all next to one another? (3)

13. Consider the letters

M A T H E M A T I C A L

a. How many different “words” can be formed using all the letters? (3)

b. If such a word is chosen at random then what is the probability that it does NOT start and end with the same letter? (5)

14. Alexandra has cards with the following digits on:

6	2	4	1	8

She has just one of each card and she is forming numbers using **some or all of** her cards.

For example:

2 1 OR 2 8 6 OR 1

a. How many different numbers can she form in this way? (4)

b. What is the probability that the number she forms is bigger than 5000? (4)

15. A group of 6 friends goes to the movie. Johan hopes he won't be sitting next to Pieter as they have recently had a disagreement. What is the probability that they are NOT seated next to one another? (4)

16. Nonhlanhla is investigating whether people's hot drink preference is dependent of their left or right handedness. She collects the following data:

	Prefer tea	Prefer coffee	**TOTAL**
Left handed	19	76	
Right handed	213		**1065**
TOTAL		**928**	**1160**

a. Complete the table (2)

b. Determine whether people's hot drink preference is dependent of their left or right handedness. Justify your answer with a calculation (3)

Common probability instruments

A **coin** has two sides, HEADS and TAILS each of which is equally likely – 50%

A die is a cube with six faces numbered from 1 – 6. Each is equally likely – $\frac{1}{6}$

A pack / deck of cards has 52 cards

- 4 of each suit (clubs, diamonds, hearts and spades)
- Each suit has 13 cards (A, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen and King)
- Clubs and Spades are black while hearts and diamonds are red
- Jack Queen and King are called picture cards
- The symbols are as follows:
    - Clubs - ♣
    - Diamonds - ♦
    - Hearts - ♥
    - Spades - ♠