Grade 12 Mathematics

Revision Workshop on Trigonometry, Geometry, Analytical Geometry and Calculus

6 June 2026

S·I·M·I

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NAME: ____

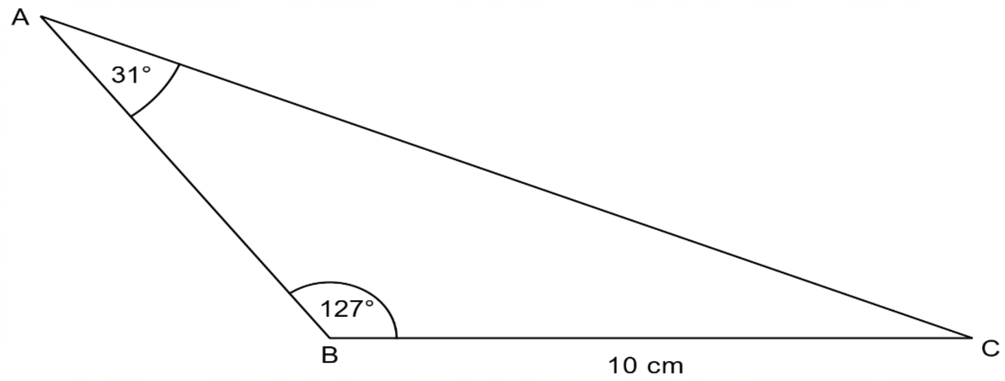
Paper 2 – Trigonometry – solution of triangles in 2 and 3 dimensions

In Δ ABC: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

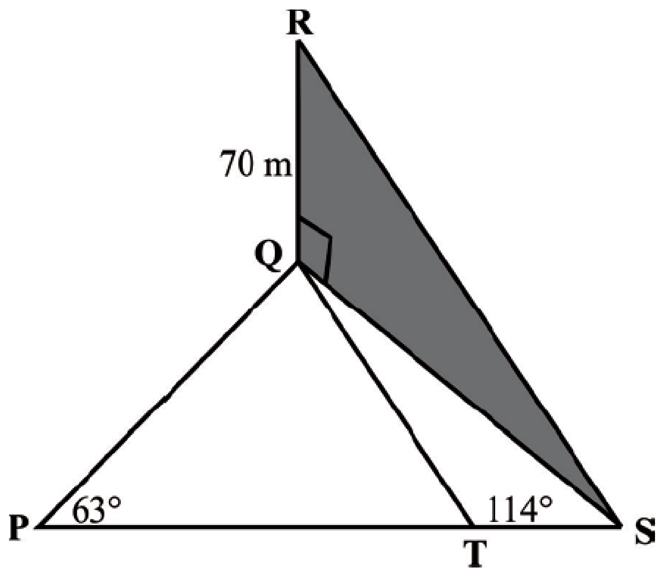
$$a^2 = b^2 + c^2 - 2bc.\cos A$$

$$\text{area } \Delta ABC = \frac{1}{2}\text{ab.sinC}$$

(a) Determine the area of the triangle below to 1 d.p.



(b) RQ is a vertical tower. P, Q, S and T are in the same horizontal plane.
PT = 90 m and ST = 54 m

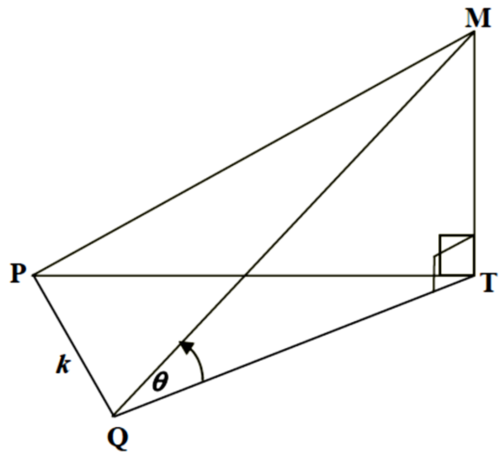


(i) Show that QT = 103 m to the nearest whole number.

(ii) Determine the angle of elevation of R from S.

(c) MT is a vertical tower. P, Q and T are in the same horizontal plane. The angle of elevation of M from Q is θ.

$PQ = k$; $PM = 2PQ$ and area $\Delta MPQ = 2k^2\sin\theta\cos\theta$



(i) Prove that $M\hat{P}Q = 2\theta$

(ii) Hence prove that $MQ = k\sqrt{1 + 8\sin^2\theta}$

Paper 2 – Geometry

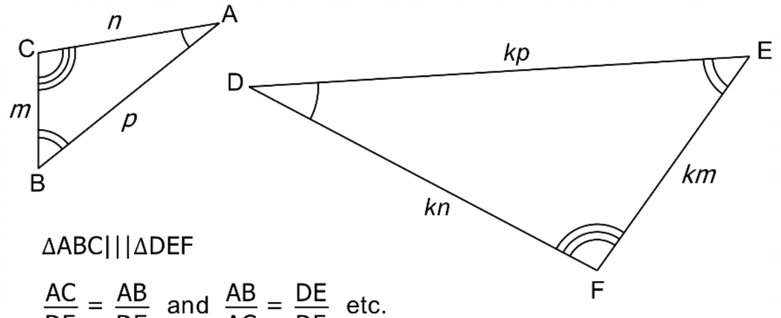
Similarity

One can prove **similarity** of triangles in one of two ways:

- Prove them equiangular (AAA)
- Prove that the corresponding sides are in proportion (sides of Δ in proportion)

In fact, both are necessary to prove shapes similar, but with triangles the one guarantees the other.

The symbol for similarity is |||. We always list similar triangles in corresponding order, following which we can pull out equal ratios as per the example below:



$\Delta ABC \,|||\, \Delta DEF$

$\frac{AC}{DF} = \frac{AB}{DE}$ and $\frac{AB}{AC} = \frac{DE}{DF}$ etc.

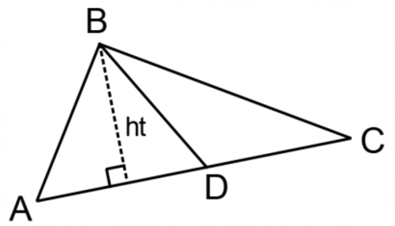
Areas (volumes) of similar shapes (solids)

Shapes that are similar with sides in the ratio $a : b$ will have **areas in the ratio** $a^2 : b^2$. In the event of 3-dimensional solids their **volumes will be in the ratio** $a^3 : b^3$

In the above example the sides are in the ratio 1 : 2 so the areas will be in the ratio $1^2 : 2^2 = 1 : 4$

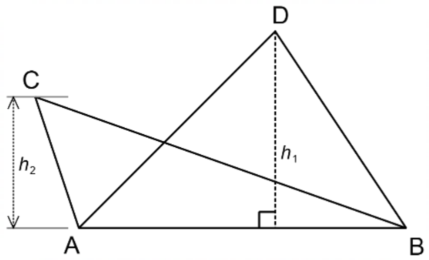
$\Delta ABC = \frac{1}{4}\Delta DEF$ or $\Delta DEF = 4\Delta ABC$

Areas of triangles sharing baes or heights



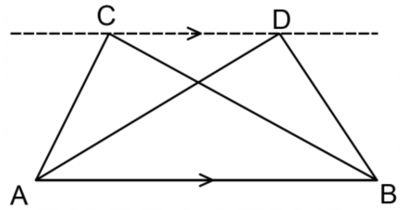
$$\frac{\Delta ABD}{\Delta ABC}=\frac{\frac{1}{2}\times AD\times ht}{\frac{1}{2}\times AC\times ht}=\frac{AD}{AC}$$

The areas of triangles with a shared / common height are in the same ratio as their bases.
(shared heights)



$$\frac{\Delta ABC}{\Delta ABD}=\frac{\frac{1}{2}\times AB\times h_2}{\frac{1}{2}\times AB\times h_1}=\frac{h_2}{h_1}$$

The areas of triangles with a shared base are in the same ratio as their heights.
(shared bases)



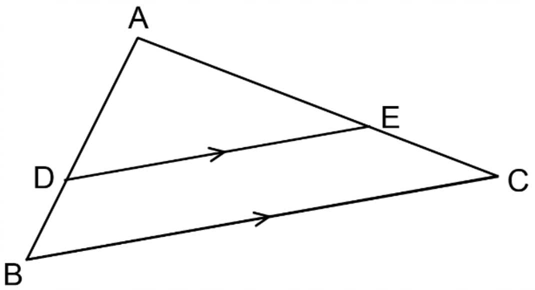
$$\Delta ABC = \Delta ABD$$

Triangles on the same base between parallel lines have equal areas.
(shared base between parallels)

From here on, what is given is in **black** and what results is in **green**!

The proportional intercept theorem

A line drawn parallel to one side of a triangle cuts the other two sides in the same proportion
(line || one side of Δ)

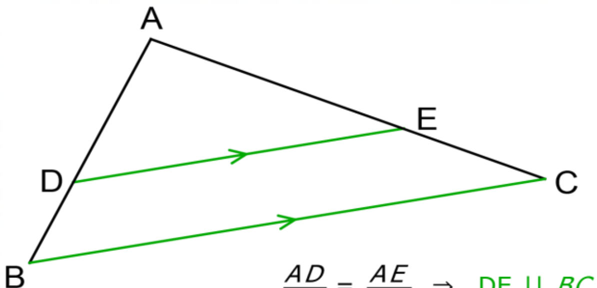


$$\frac{AD}{DB}=\frac{AE}{EC}$$

or

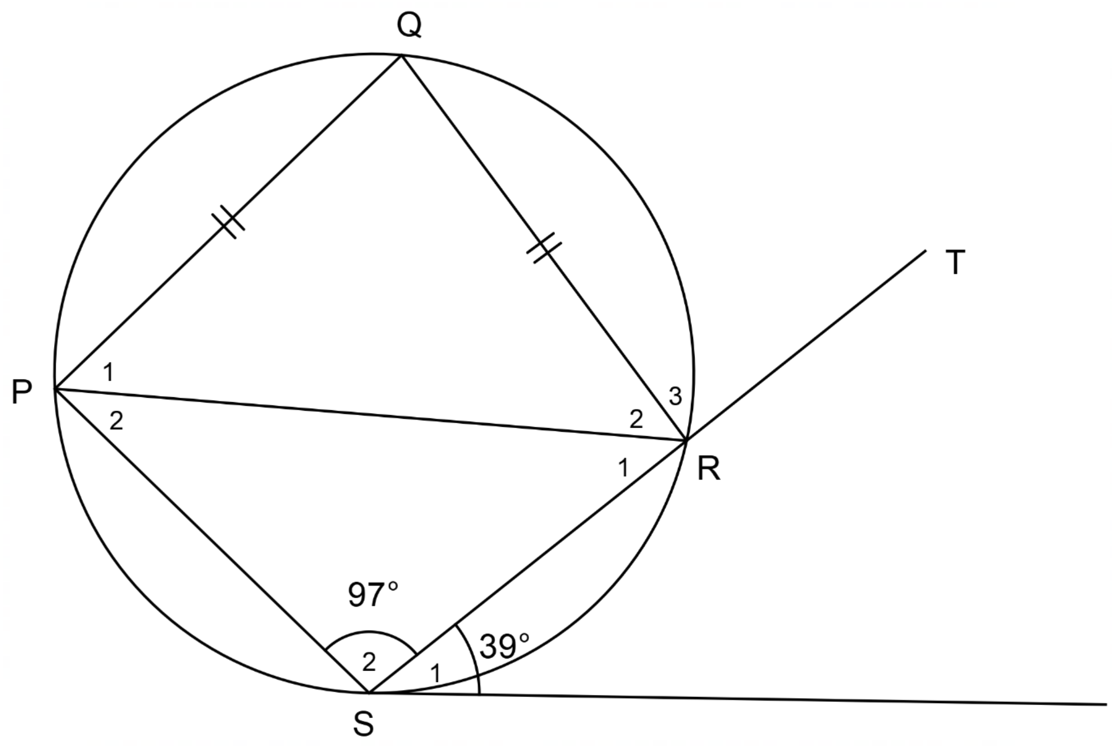
$$\frac{DB}{AB}=\frac{EC}{AC}$$

The converse is true. In other words, if a line cuts two sides of a triangle in the same proportion, then it will be parallel to the third side (line divides two sides of Δ in prop)



$$\frac{AD}{DB}=\frac{AE}{EC}\Rightarrow DE\parallel BC$$

(a) In the diagram, PQRS is a cyclic quadrilateral. There is a tangent to the circle at S and chord SR is produced to T. $PQ = QR$, $\hat{S}_1 = 39°$ and $\hat{S}_2 = 97°$



Determine, with reasons, the size of the following angles:

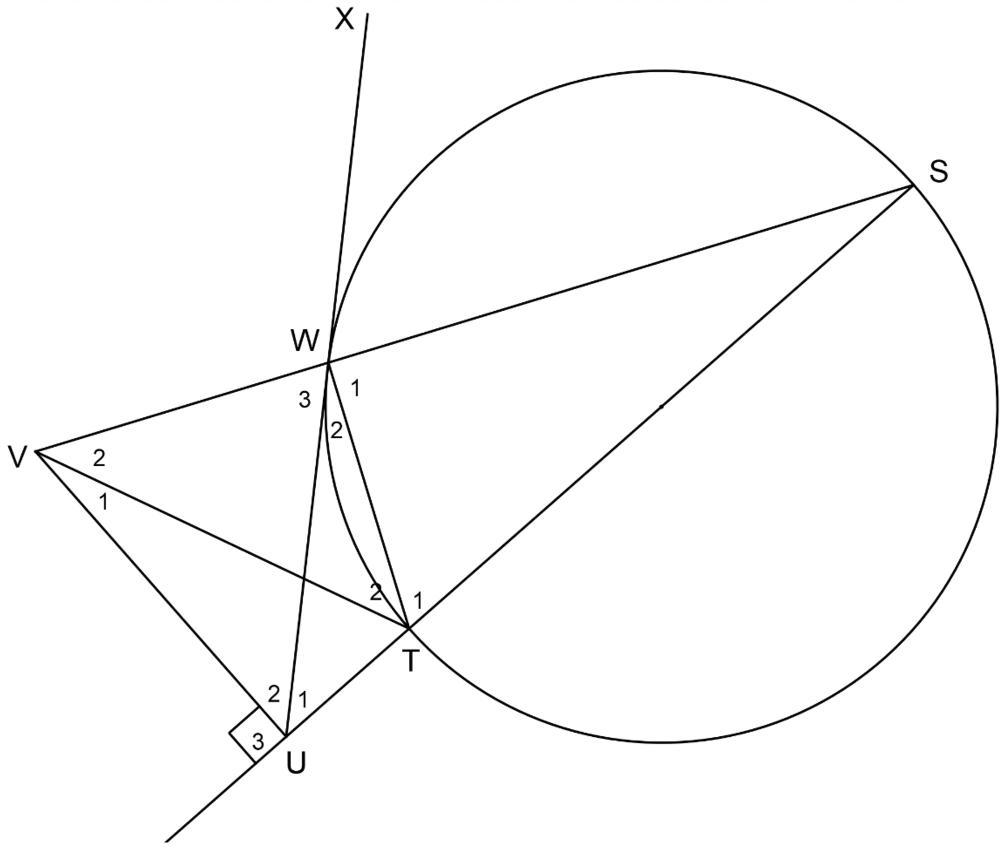
(i) $\hat{Q}$

(ii) $\hat{R}_2$

(iii) $\hat{P}_2$

(iv) $\hat{R}_3$

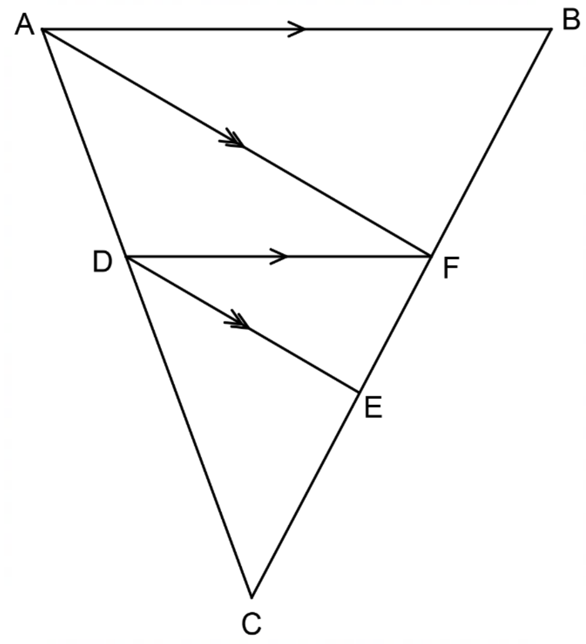
(b) In the diagram the **diameter** ST is produced to U. XWU is a tangent to the circle at W and chord SW is produced to V. $VU \perp US$.



(i) Prove that WVUT is a cyclic quadrilateral.

(ii) Prove that VU is a tangent to the circle passing through V, T and S.

(c) In the diagram below $AB \| DF$ and $AF \| DE$. $CF : FB = 3 : 2$



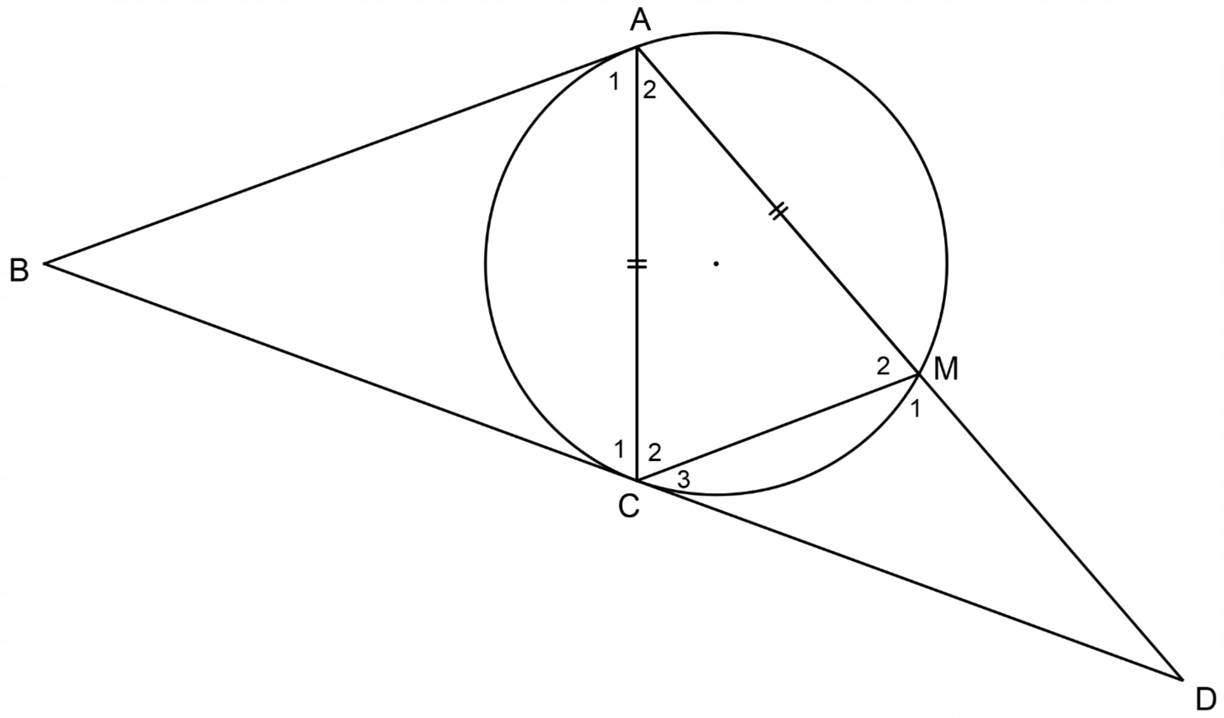
Determine, with reasons:

(i) $AD : DC$

(ii) area ΔAFD : area ΔDFC

(iii) $EF : FB$

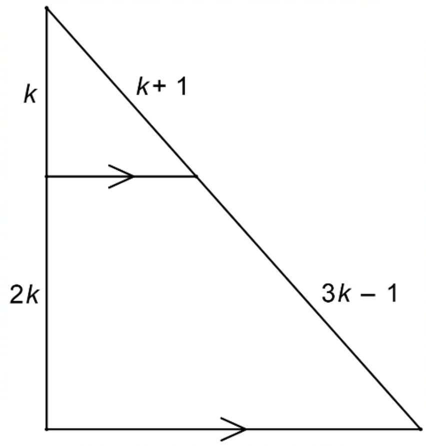
(d) AB and BC are tangents. AC = AM.



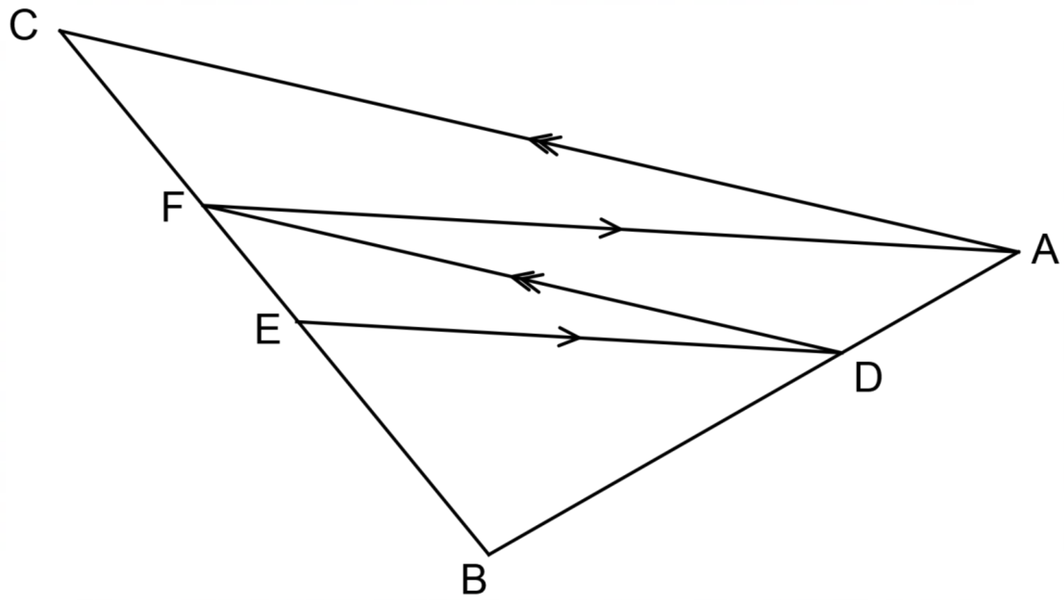
(i) Prove that $CD^2 = DM.AD$

(ii) Prove that $\dfrac{DM}{MA} = \dfrac{DC}{CB}$

(e) Determine the value of k giving a reason.



(f) In the diagram below $DF||AC$ and $AF||DE$. If $AD : AB = 1 : 3$ and BC = 120mm then determine, with reasons, EF to 2 d.p. if necessary.



Paper 2 – Analytical Geometry

Consider two points: $(x_1 ; y_1)$:

The **distance** between them: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The **midpoint**: $\left(\frac{x_1 + x_2}{2} ; \frac{y_1 + y_2}{2}\right)$

The **gradient** of the line joining them is: $m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

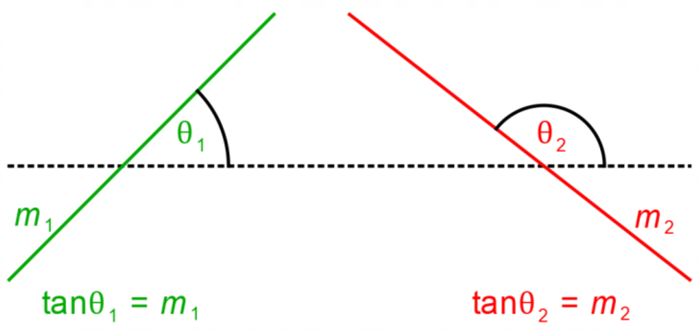
Remember that lines which slope **uphill** have positive gradients while lines which slope **downhill** have negative gradients!

Parallel lines have equal gradients while **perpendicular** lines have gradients that are the negative reciprocal of one another.

They multiply together to give –1. examples: $\frac{2}{5}$ and $-\frac{5}{2}$ or 3 and $-\frac{1}{3}$.

Collinear points are points which lie on the same straight line. No matter which two one uses they will generate the same gradient!

The **angle of inclination** of a line is measured anti-clockwise from the positive x-axis.
The gradient of a line is equal to the tan of the **angle of inclination**. $m = \tan\theta$



Lines

We need to be able to find the equation of a straight line. We can do this if we have the gradient of the line and a point on it. We can use either of these two formulae:

$y = mx + c$ or $y - y_1 = m(x - x_1)$

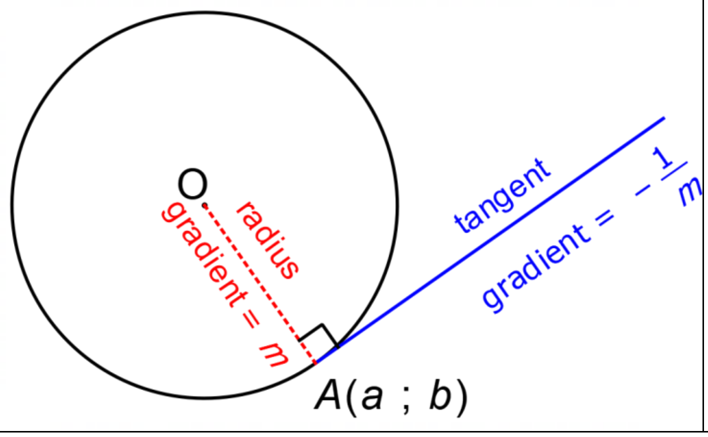
Circles $(x-a)^2+(y-b)^2=r^2$ is a circle with centre $(a\,;\,b)$ and radius r

When a circle is not given in this standard form, we complete the square to put it in standard form as per the following example:

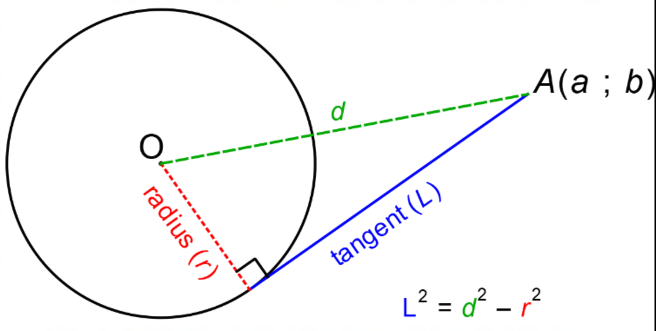
$$x^2-6x+y^2+8y=11$$
$$\therefore (x-3)^2+(y+4)^2=11+9+16$$
$$\therefore (x-3)^2+(y+4)^2=36$$

so, a circle with centre (3;-4) and radius 6

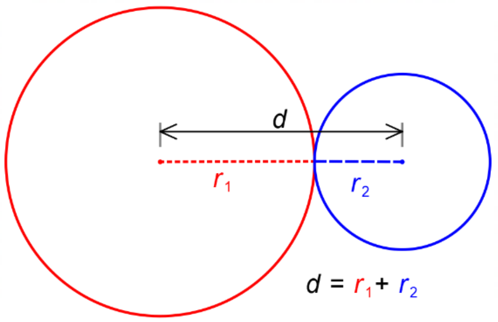
To find the **equation of a tangent to a circle** at a given point we use the fact that the tangent is perpendicular to the radius at the point of contact. We can typically find the gradient of the radius and then use this fact to find the gradient of the tangent.

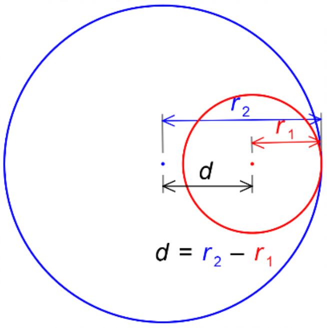


We can also use Pythagoras to find the **length (L) of a tangent to a circle from a given point**.



We can **determine whether two circles touch, intersect or miss** by looking at the sum / difference of their radii and the distance between their centres.





if $d < r_1 + r_2$ then the circles will **cut**

if $d = r_1 + r_2$ then the circles will **touch**

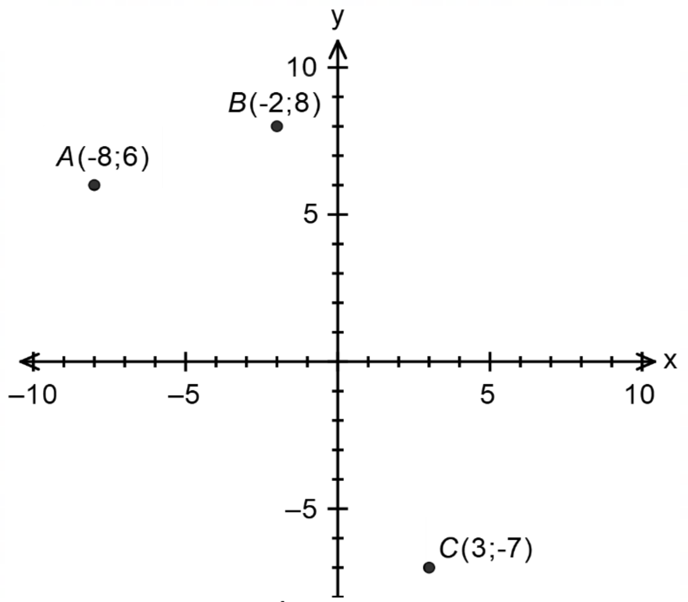
if $d > r_1 + r_2$ then the circles will **miss**

if $d < r_2 - r_1$ then the circles will **miss**

if $d = r_2 - r_1$ then the circles will **touch**

if $d > r_2 - r_1$ then the circles will **cut**

(a) Consider the points as plotted below:



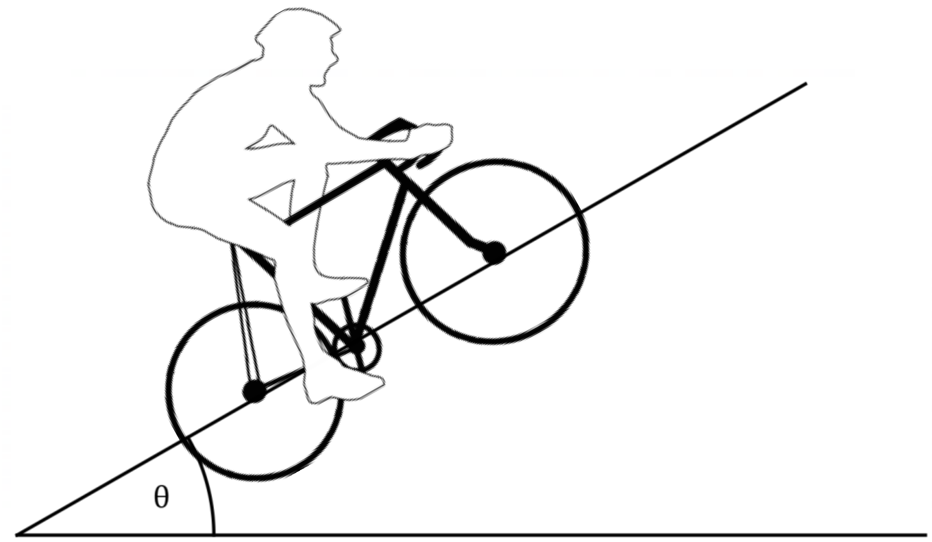
(i) Show that $A\hat{B}C = 90°$

(ii) Hence, or otherwise, determine the area of ΔABC

(iii) Find the mid-point of line AB

(iv) Find point D if ABCD (in that order) is a parallelogram.

(b) Consider the sketch of a man riding up a hill. The equation of the straight line representing the road is $y = \frac{x}{\sqrt{3}}$ and the equation of the circle representing the rear wheel is $x^2 - 8x + y^2 - 4y = 1004$.

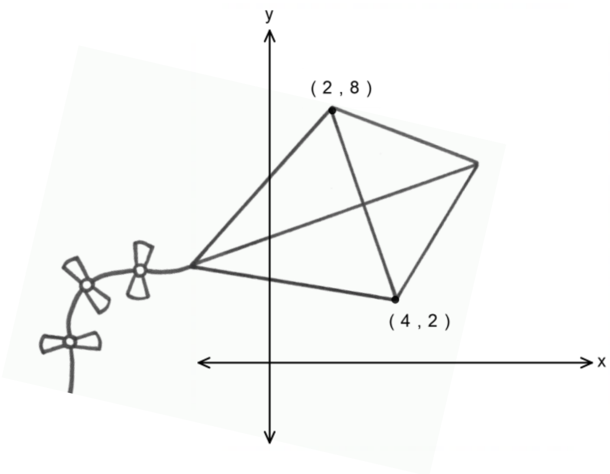


(i) Determine the angle the road makes with the horizontal

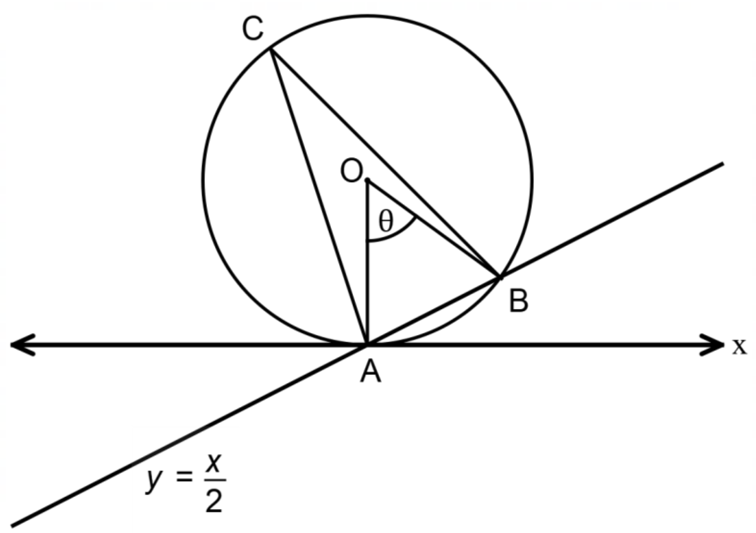
(ii) Assuming that the units are in cm, determine the diameter of the rear wheel, in cm.

(c) *"The longer diagonal of a kite bisects the shorter diagonal at 90°"*

Use the above statement to find the equation of the longer diagonal of this kite.



(d) In the diagram below the x-axis is a tangent.to the circle and the line $y = \frac{x}{2}$ is a secant to the circle.



Determine θ correct to 2 d.p. showing working and giving reasons.

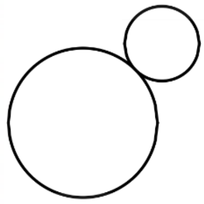
(e) Consider the two circles:

Circle ① : $x^2 + y^2 - 6x - 4y = -5$

and

Circle ② : $(x - 6)^2 + (y - 5)^2 = p$

Circle ②



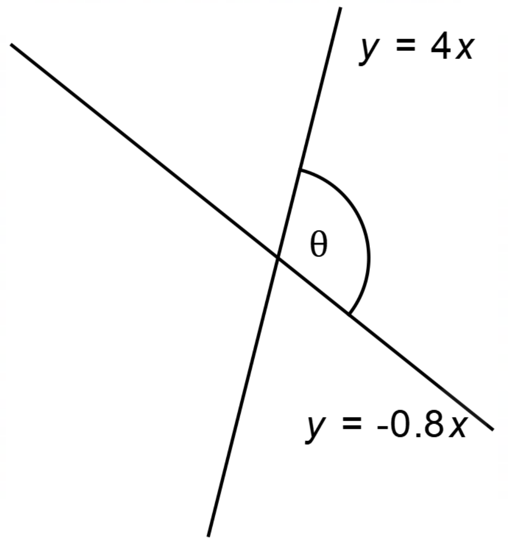
Circle ①

(i) Find the centre and radius of Circle ①.

(ii) Hence, or otherwise, find the possible value(s) of p if it is known that Circle ① and Circle ② touch externally as shown.

(iii) Determine the point at which the two circles touch.

(f) Find θ (2 d.p.) in the diagram below.



(g) Consider the circle $x^2 + y^2 + 6x - 10y = 31$

(i) Determine the equation of the tangent at the point $A(4;1)$

(ii) Suppose a tangent (not the one you found above) is drawn from $B(12;-4)$ to a point of contact with the circle, D. Determine the length of BD.

Paper 1 – Calculus

To find the average gradient between two points we use our gradient formula:

$$\text{gradient} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The derivative of a function gives its **instantaneous gradient**.

The process of finding a derivative is called **differentiation**.

The process of d**ifferentiation by first principles** is important.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Remember:

- to have limit on every line where h occurs but never without h
- Never suggest that $h = 0$ – it **cannot**!
- = sign goes on left of limit!
- use brackets when you no longer have a fraction – the limit applies to all terms

Unless asked to use first principles we use our **rules for differentiation**. They are given in English and Maths below:

The derivative of x raised to a power is the power multiplied by x raised to the power reduced by one.	$\frac{d}{dx} x^n = nx^{n-1}$
The derivative of a constant is zero. The line $y = c$ is horizontal with a gradient of zero.	$\frac{d}{dx} c = 0$
The derivative of a sum / difference is equal to the sum / difference of the derivatives	$\frac{d}{dx} [f(x) \mp g(x)] = f'(x) \mp g(x)$
The derivative of a constant times a function is equal to the constant times the derivative of the function	$\frac{d}{dx} [c.f(x)] = c.f'(x)$

Before we can apply our rules, we need to ensure that the expression is **flat** – no brackets, fractions, or surds.

- Products can be expanded
- Surds can be removed using $\sqrt[a]{x^b} = x^{\frac{b}{a}}$ – e.g. $\sqrt[3]{x^4} = x^{\frac{4}{3}}$
- Fractions can be split and "flattened" – e.g. $\frac{x^2 + 3x^7}{2x^5} = \frac{1}{2} x^{-3} + \frac{3}{2} x^2$

The following **<u>notations</u>** are important:

if $y = \ldots$ then $\frac{dy}{dx}$ represents the first derivatve and $\frac{d^2y}{dx^2}$ the second derivative, both with respect to x

if $f(x) = \ldots$ then $f'(x)$ represents the first derivative and $f''(x)$ the second derivative

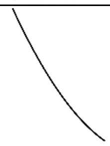
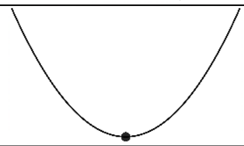
$D_x[f(x)]$ and $\frac{d}{dx}(f(x))$ both mean the derivative of $f(x)$ with respect to x

We need to pay careful attention to notation – never suggest that the function is the derivative or vice versa!

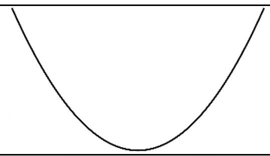
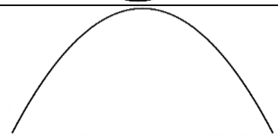
To **optimize** (make as big / small as possible) a quantity we need to get an expression for it involving **only one** variable. Then we need to differentiate with respect to that variable, equate to zero and solve. Make sure you have answered the question as sometimes we are asked for the value of the variable that optimizes the quantity, sometimes the optimum quantity and sometimes both!

A derivative is a **rate of change**. If it is positive, then the function / quantity is increasing and if negative then the function / quantity is decreasing.

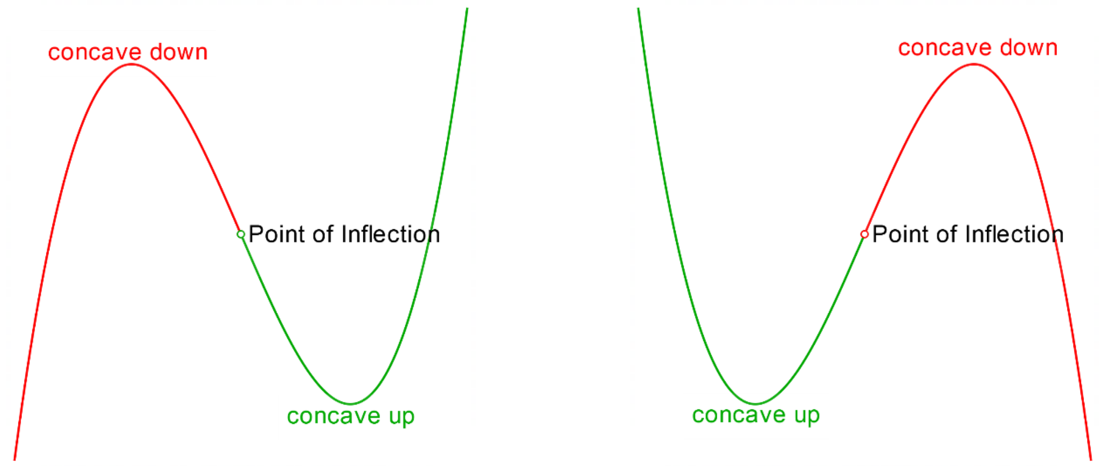
The first derivative tells us about the **rate of change of the function**. If the first derivative is:

positive the graph is increasing, going uphill from left to right	
negative the graph is decreasing, going downhill from left to right	
zero then the graph is stationary	

The second derivative tells us about the **rate of change of the first derivative / gradient.** If the second derivative is:

positive, then the gradient is increasing, and the graph is **concave up** – like a cup	
negative, then gradient is decreasing, and the graph is **concave down** – like a frown	

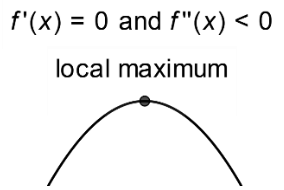
At a point of inflection, the graph changes **concavity**. The graph goes from concave down to concave up or vice versa. The gradient goes from decreasing to increasing decreasing or vice versa.



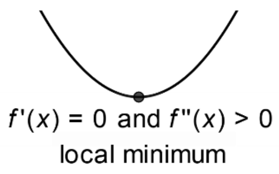
At a **point of inflection,** the **second derivative is zero**. However, the second derivative can be zero without having a point of inflection (e.g., $y = x^4$ at $x = 0$). To confirm that we have a point of inflection we should check that the second derivative changes sign on either side of that point.

If we know the first derivative is zero, we can calculate the second derivative to determine the **nature of our stationary point**:

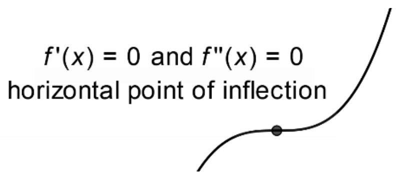
- if the second derivative is negative this is a **maximum turning point.**



- if the second derivative is positive this is a **minimum turning point**.

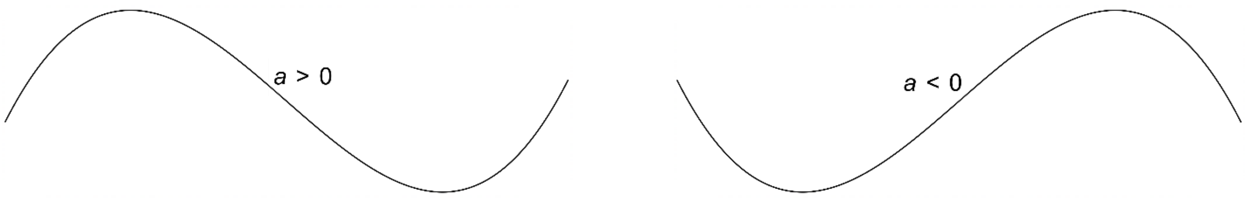


- if the second derivative is zero this is a **horizontal point of inflection**.



Sketching a cubic function $y = ax^3 + bx^2 + cx + d$

- The leading coefficient (a) gives us a sense of the shape.



- find the y-intercept by letting $x = 0$
- find the x-intercept(s) by letting $y = 0$
- find the x-values of the stationary points by letting $\frac{dy}{dx} = 0$. We substitute these values into the original equation to find their corresponding y values.

Given an x-value we find the **equation of the tangent** to a curve by using $y = mx + c$. We get the y value of the point from the function itself and the gradient (m) from its derivative evaluated at that x-value.

Differentiating displacement (distance) gives velocity (speed) and differentiating velocity (speed) gives acceleration.

(a) Differentiate $f(x) = x^2 + 3x + 2$ by first principles

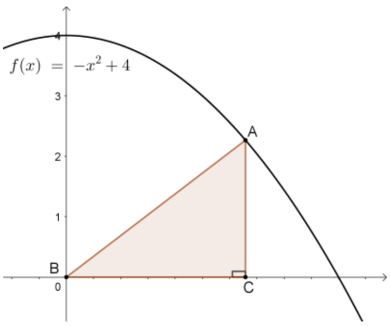
(b) Determine the following:

(i) $D_x\left[(x + 2)^2\right]$

$g'(x)$ if $g(x) = \dfrac{x^2 - 4}{6 - 3x} + \sqrt{x}$

(c) Determine if $f(x) = x^3 + 5x^2 - 1$ is increasing, decreasing or stationary when $x = -1$

(d) A right-angled-triangle ABC is formed by with one corner (B) on the origin and another (A) on the curve $f(x) = -x^2 + 4$ as shown.



(i) If C is the point $(k;0)$ then show that

Area $\Delta ABC = -\frac{k^3}{2} + 2k$

(ii) Hence, determine the maximum area of the triangle if $0 < k < 2$

(e) Determine for which value(s) of x the function: $f(x) = x^3 - x^2 - x - 1$ is concave down?

(f) If the function $g(x) = x^3 + ax^2 + 3x - 1$ has a stationary point at $x = 1$ then find a

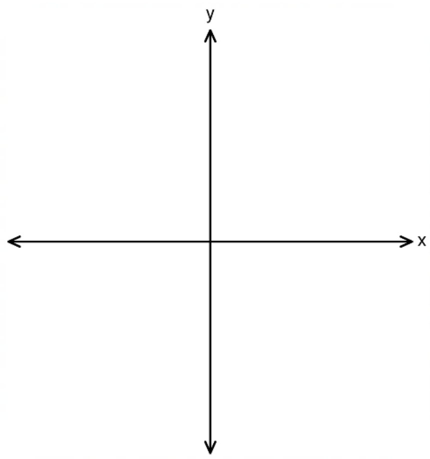
(g) At which x-value(s) is/are the tangent(s) to the function $y = x^3 - 2x^2 + x$ parallel to the line $2y - 10x + 8 = 0$?

(h) Find the equation of the tangent to the curve $y = x^2 + 4x - 1$ which is perpendicular to the line $2y + x = 4$

(i) Find a and b if $g(x) = ax^3 + bx^2 - 4x - 3$ has a turning point at (-2;9)

(j) Given: $f(x) = ax^3 + bx^2$

1) If $f'(0) = f'(2) = 0$ and $f(2) = 8$ and $f(x)$ is increasing for $x \in (0;2)$ then draw a rough sketch of the graph of $f(x)$



2) Solve for a and b and hence write down the equation of $f(x)$