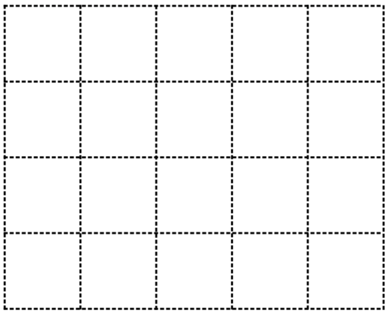
Module 6 : CALCULUS

Differential Calculus – The Study of Rates of Change

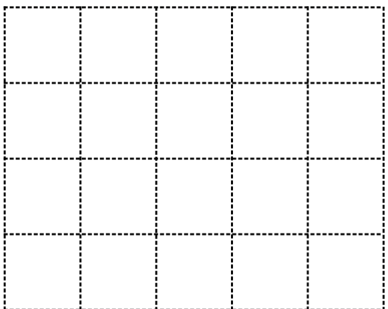
Differential Calculus is all about determining rates of change.

A **gradient** is a rate of change $\frac{\Delta y}{\Delta x}$. For example:

a gradient of $\frac{2}{5}$ means that y changes (increases) by 2 for a change (an increase) of 5 in x



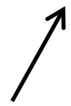
a gradient of $-3 = \frac{-3}{1}$ means that y changes (decreases) by 3 for a change (an increase) of 1 in x



Let's start by reminding ourselves about the process of finding gradients of straight lines.

$$\text{gradient} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Remember too that lines which slope uphill (left to right) have a positive gradient



while lines which slope downhill have a negative gradient!

Let's practise finding some gradients:

1. Determine the gradient of the line segment which joins:

   a. (2;3) and (7;19)

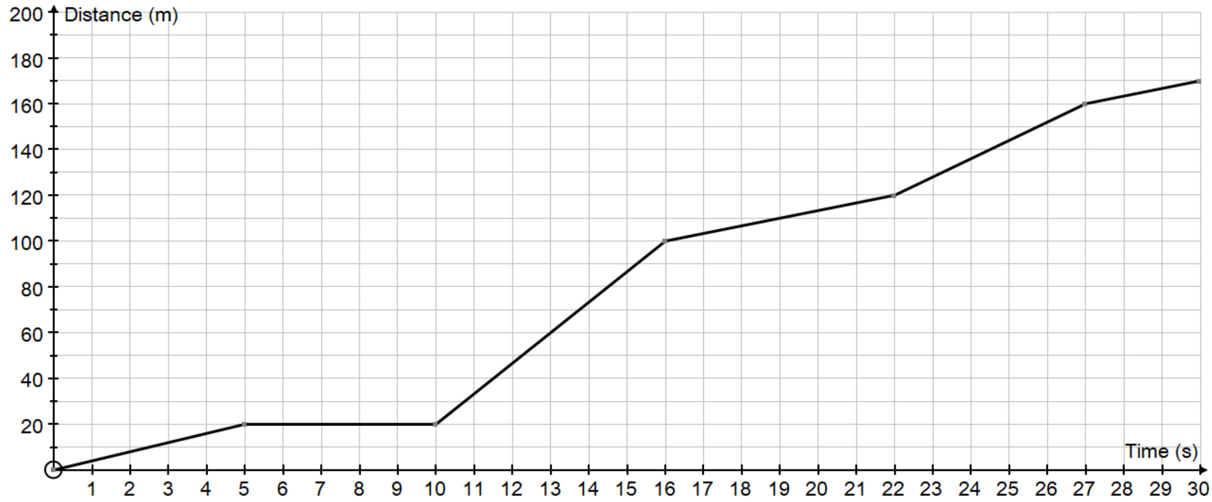
   b. (3;-2) and (5;-9)

   c. (-2;6) and (5;-1)

   d. (3;-1) and (5;3)

2. Let's consider some real-world examples of rates of change:

a. The graph below shows the distances (in metres) covered over time (in seconds) by an athlete in training.



In answering the questions below be sure to give the correct units with your answers!

i. What was his average rate of change with respect to time (speed) over the whole session?

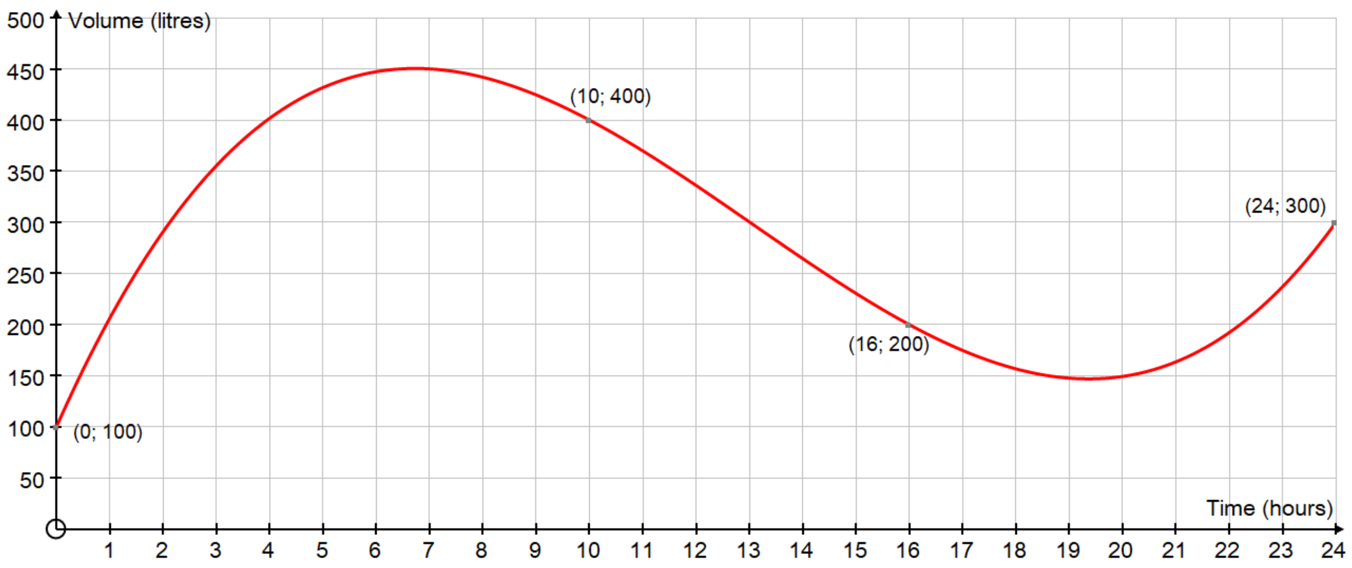
ii. During what time interval was he going the fastest and how fast was that?

iii. When was he stationary?

iv. What was his instantaneous speed at 20 seconds?

v. What was his average speed between the 10th and the 22nd seconds?

b. The level of water in a tank over a 24-hour period is shown in the graph below:



In answering the questions below be sure to give the correct units with your answers!

i. What was the average rate of change of volume with respect to time over the 24-hour period? Did the volume increase or decrease over that time?

ii. What was the average rate of change of volume with respect to time between the 10th and the 16th hours? Did the volume of water increase or decrease over that time?

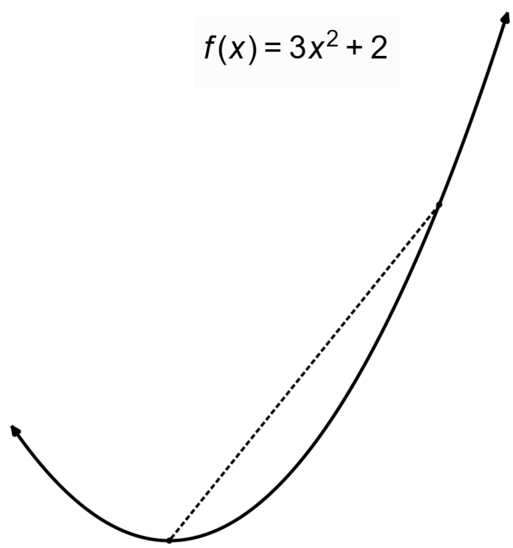
iii. After how many hours was the tank at its fullest – approximately?

3. When we use two points on a curve to find the gradient, we are finding the **average gradient** of the curve between these two points.

a. Find the average gradient of the function

$f(x) = 3x^2 + 2$ between the points where $x = 0$ and where $x = 2$

Note that we will first need to first work out the y-values and then work out the gradient using the two $(x;y)$ points:

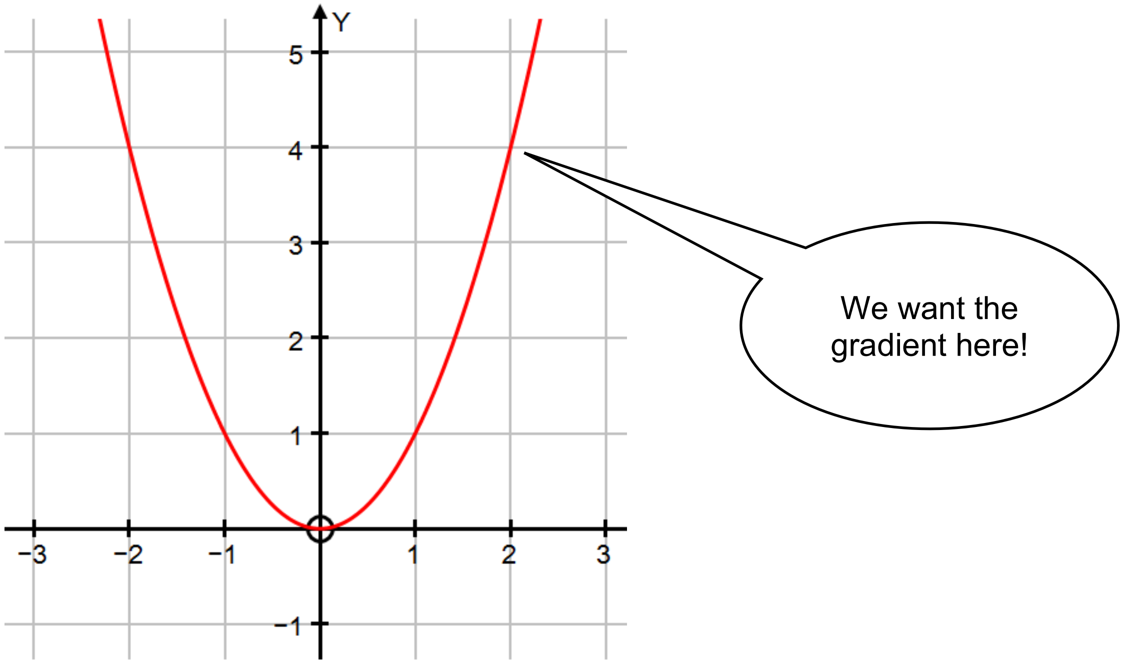


b. $y = -x^2 + 1$ between the points where $x = -1$ and where $x = 2$

c. $f(x) = x^2$ between the points where $x = -2$ and where $x = 1$

4 Now consider the challenge of finding the gradient of the curve $y = x^2$ at the point where $x = 2$

This is called an **instantaneous gradient** as the gradient is changing all the time as we move along the curve.



What we will do is investigate what happens to the gradient as we use two points, the point (2;4) and another point which gets increasingly close to that point.

Complete the following table:

First point	**Second Point**	**Gradient**
(2;4)	(3;9)	
(2;4)	$\left(2\frac{1}{2}\ ;\ 6\frac{1}{4}\right)$	
(2;4)	(2,2 ; 4,84)	
(2;4)	(2,1 ; 4,41)	
(2;4)	(2,01 ; 4,0401)	
(2;4)	(2,001 ; 4,004001)	

What do you notice?

There seems to be a limit!

As the second point approaches the point $(2;4)$, the gradient seems to approach 4. In fact, the gradient of the curve $y = x^2$ at the point where $x = 2$ is 4.

Let's look at a simulation of this.

A brief digression on limits!

When we write:

$$\lim_{x \to 1}(x^2 + 3x)$$

we are considering how $x^2 + 3x$ behaves when x gets close to 1.

The answer is obviously 4.

Notice, we are not concerned with how $x^2 + 3x$ behaves when x **actually** equals 1 although, in this instance it is the same.

Notice, also that $\lim_{x \to 1} x^2 + 3x$

This is like the difference between and

Now let's consider a more interesting example:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

Notice that, in the above example x cannot equal 2. However, it can assume values as close to 2 as we like. The closer x gets to 2, the closer $\frac{x^2 - 4}{x - 2}$ gets to

We write $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} =$

Note that the point (2;4) does not actually lie on the original function. If we had to graph it is a straight line with a "hole" in it at (2;4). This is called a discontinuity although it is not a term we need to know for core maths.

You try this one:

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4}$$

Some really **important** things:

1. The word limit must be on every single line except the last where the limit has been calculated

2. The equals signs must be to the left of the word limit and not to the right

3. If there is more than one term, then we must use brackets

4. NEVER do anything to suggest that *x* is equal to the value where we are trying to find the limit. *x* need not and often cannot equal that value.

5. We are interested in how the function behaves **near** the value where we are trying to find the limit NOT **at** the value.

Determine:

1. $\lim_{x \to 1} \frac{x^2 - 4x + 3}{2x - 2}$

2. $\lim_{t \to 2} \frac{t^2 + 2t - 8}{t - 2}$

3. $\lim_{p \to 1} \frac{p^2 + 5p - 6}{p - 1}$

4. $\lim_{h \to 0} \frac{h^2 + 3h}{h}$

A brief reminder of function notation!

Suppose we have the function: $f(x) = 2x^2 + 5x - 7$

Then $f(x)$ represents the value of the function f at the point x. This is the same as the y-value and the point $(x\ ;\ f(x))$ lies on the function.

$f(0) =$

And $(\quad ; \quad)$ is a point on the function.

$f(2) =$

And $(\quad ; \quad)$ is a point on the function.

$f(-3) =$

And $(\quad ; \quad)$ is a point on the function.

$f(a) =$

And $(\quad ; \qquad\qquad\qquad)$ is a point on the function.

$f(x + h) =$

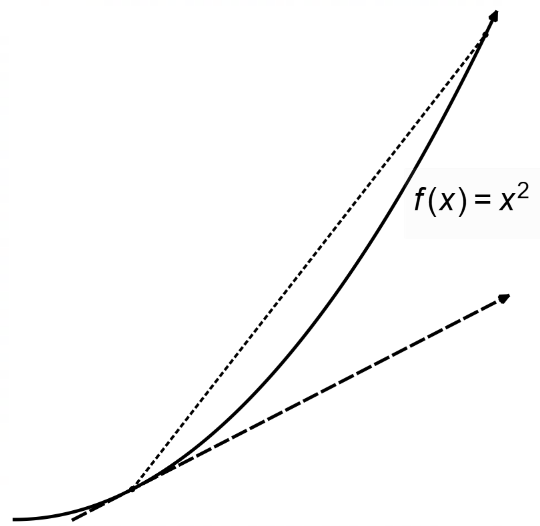
And $(\qquad ; \qquad\qquad\qquad\qquad\qquad)$ is a point on the function.

Back to the problem of finding the **instantaneous gradient** of a curve at a point!

Let's try something more general. For now, we will stick with the same function $f(x) = x^2$

Suppose our first point is the point $(x;x^2)$ and the second point is h units further away at $\left(x+h\ ;\ (x+h)^2\right)$

We are trying to find the **instantaneous gradient** of the curve at the point $(x;x^2)$. This is the same as the gradient of the **tangent** to the curve at that point (shown as a dashed line)



In the space below, determine the **average gradient** using these two points. This is the gradient of the dotted line. Your answer will obviously be in terms of x and h. Simplify your answer.

Now suppose the two points get closer and closer together! In other words, determine the limit of your gradient as h approaches zero!

So, the function $f(x) = x^2$ has a gradient which is always changing but always equal to twice the x-value.

We use the notation $f'(x)$ to indicate the gradient of $f(x)$

So, if $f(x) = x^2$ then

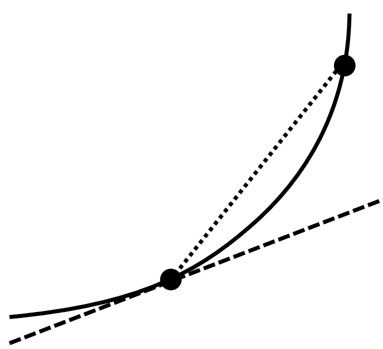
Let us look at a simulation of this.

In tests and exams, we are given the formula: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

This is just the **average gradient** (represented by a dotted line) of the curve of f between the points:

$(x\ ;\ f(x))$ and $(x + h\ ;\ f(x + h))$

with $h \to 0$ to give us the **instantaneous gradient** (represented by a dashed line) at the point $(x;f(x))$



average gradient $= \dfrac{f(x+h) - f(x)}{x + h - x}$

average gradient $= \dfrac{f(x+h) - f(x)}{h}$

instantaneous gradient $= f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Now see if we can use this formula to work out a rule for the gradient on the curve $f(x) = 2x^2 + x$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

DIFFERENTIATION RULES!

The process of finding a gradient function (a **derivative**) is called **differentiation**.

So far we have done this the "long way" called **differentiation by first principles**.

While one needs to be able to do this, there are some quick rules to help us!

Remember $\frac{d}{dx}(\quad)$ means **find the derivative (gradient)** of the expression in the brackets.

$\frac{d}{dx}c = 0$ — the derivative (gradient of a constant $y = c$) is zero.

$\frac{d}{dx}x^n = nx^{n-1}$ — to differentiate a power of x we bring the power to the front and reduce it by 1.

$\frac{d}{dx}[cf(x)] = cf'(x)$ — the derivative of a constant times a function is the constant times the derivative of the function.

$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$ — the derivative of a sum is the sum of the derivatives.

$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$ — the derivatives of a difference is the difference of the derivatives.

1. Differentiate the following paying careful attention to notation:

a. $y = (3x+2)(x-1)$

b. $g(x) = \dfrac{x^2 - x - 6}{2x - 6}$

c. $\dfrac{d}{dx}\,\dfrac{3x+6}{3x^4}$

d. $D_x\left[\sqrt{x} + \sqrt{2}x + \sqrt{2}\right]$

e. $f(x) = -2x^2 + 7x + 3$

f. $\dfrac{d}{dx}\,\dfrac{x^5 + x^2}{x^3}$

g. $f(x) = \dfrac{3x^2 + 5x}{x^2}$

h. $y = \dfrac{x^2 - x - 6}{4x - 12}$

i. $D_x\left[\dfrac{6x^3 + 4x^8}{2x^5} + \sqrt[4]{x^3}\right]$

j. $g(x) = \dfrac{1}{5x^2}$

k. $y = \sqrt{2}x + \sqrt{4x} + \sqrt{4}$

l. $f(x) = \dfrac{7x^3 + 5x}{x^3}$

m. $D_x\left[(x^2+1)(x-3)\right]$

n. $\dfrac{d}{dx}\left((x-1)(x^2+x+1)\right)$

2. Find the gradient of the given functions at the given points:

a. $f(x) = 3x^2 - 2x$ where $x = 2$

b. $y = x^3 - 5$ where $x = 1$

c. $g(x) = \dfrac{1}{x}$ where $x = -1$

d. $f(x) = \sqrt{x}$ where $x = 4$

2. Now try completing these factorizations by filling in the missing bracket:

a.

$$2x^3 + x^2 - 8x - 4$$
$$= (x-2)(\qquad\qquad\qquad\qquad)$$

b.

$$2x^3 - 15x^2 - 36x + 81$$
$$(x+3)(\qquad\qquad\qquad\qquad)$$

c.

$$8x^3 + 4x^2 - 10x - 2$$
$$= (x-1)(\qquad\qquad\qquad\qquad)$$

d. Look out with this one!

$$8x^3 - 4x^2 - 130x - 1\cancel{6}3$$ should be -45. Sincere apologies

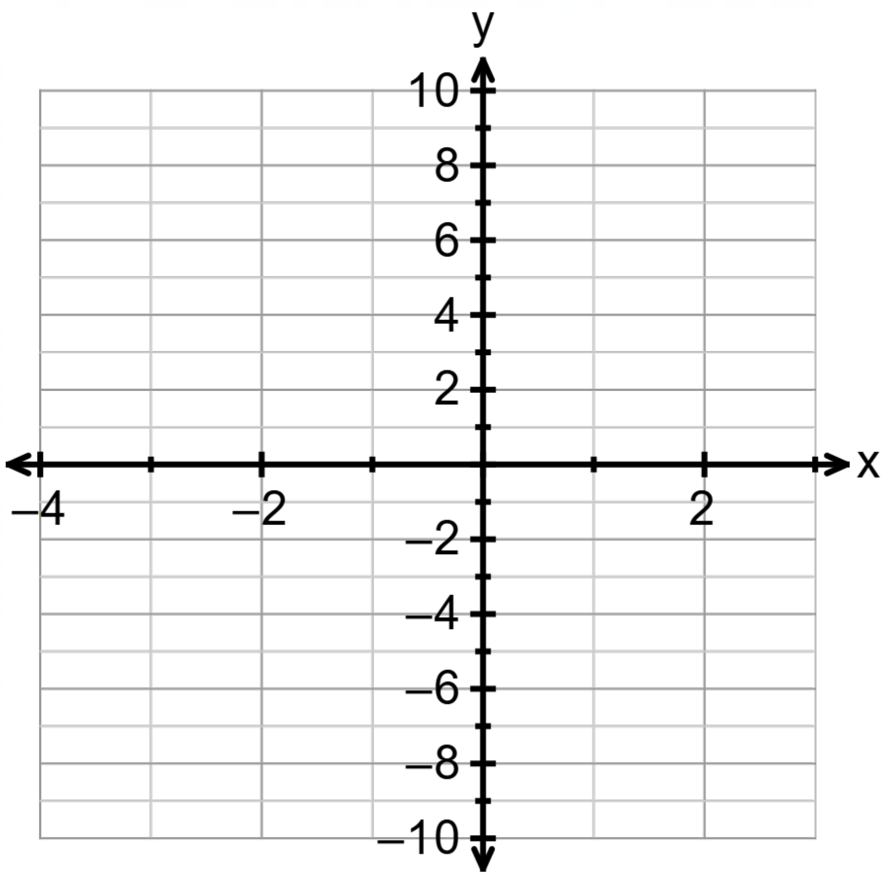
$$= (2x-9)(\qquad\qquad\qquad\qquad)$$

e. And this one!

$$3x^3 + 5x^2 - 26x + 8$$
$$= (3x-1)(\qquad\qquad\qquad\qquad)$$

3. Sketch each of the following functions, showing all points of interest. You should also show your working!

a. $f(x) = 2x^3 + 5x^2 - 4x - 3$



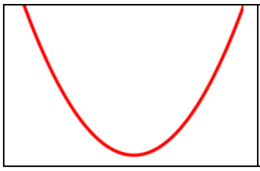
THE SECOND DERIVATIVE

In the same way that the **first derivative** tells us about the behaviour of the function (if positive, the function is increasing, if negative then decreasing), the **second derivative** (the derivative of the derivative) tells us about the rate of change of the gradient.

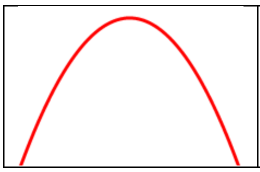
If the function is given as $y = \ldots$ then the second derivative is denoted by $\frac{d^2y}{dx^2} = \ldots.$ Some textbooks also use $y'' = \ldots$

If the function is given as $f(x) = \ldots$ then the derivative is denoted by $f''(x) = \ldots$

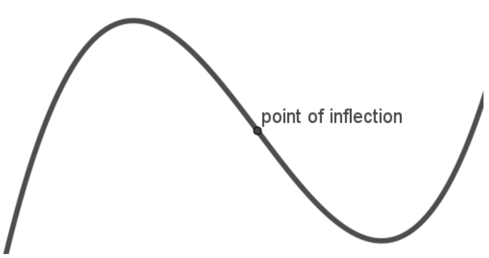
If the **second derivative is positive** at a point, then the **gradient is increasing** as we move through that point from left to right. The function would appear like this:

	We say that the function is **concave up**, like a **cup**

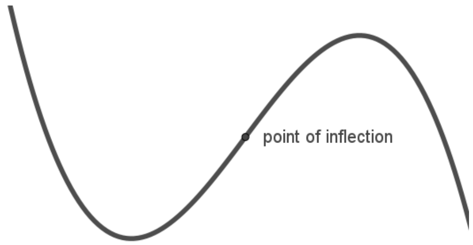
If **the second derivative is negative** then the **gradient is decreasing** as we move through that point from left to right. The function would appear like this:

	We say that the function is **concave down**, like a **frown**

If the **second derivative is zero,** then the **concavity of the graph is changing** at that point. In other words, the graph is going from concave down to concave up or vice versa. We call this a **point of inflection**.



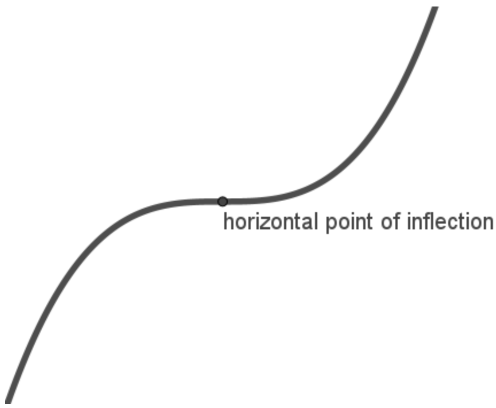
Function changes from concave down to concave up at the point of inflection



Function changes from concave up to concave down at the point of inflection

We can find the x-coordinate of a point of inflection by letting the second derivative equal zero and solving for x. We would obviously find the y-coordinate of the point by substituting the x-value we have found back into the original function.

The gradient will not typically be zero at a point of inflection, but it can be. If it is then we have a **horizontal point of inflection**.



The function changes from concave down to concave up at this horizontal point of inflection.

We can use the second derivative **to test the nature of a stationary point**

Suppose we have a point (c ; f(c)) on a curve where $f'(c) = 0$

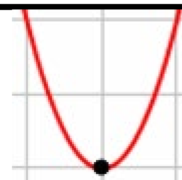
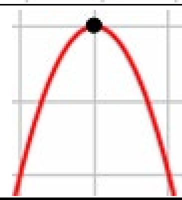
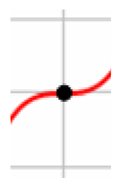
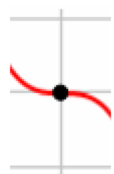
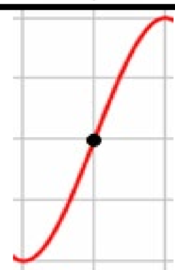
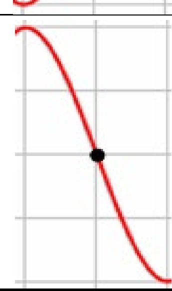
if $f''(c) < 0$ then (c ; f(c)) is a MAXIMUM turning point

if $f''(c) > 0$ then (c ; f(c)) is a MINIMUM turning point

if $f''(c) = 0$ then (c ; f(c)) is a HORIZONTAL POINT OF INFLECTION

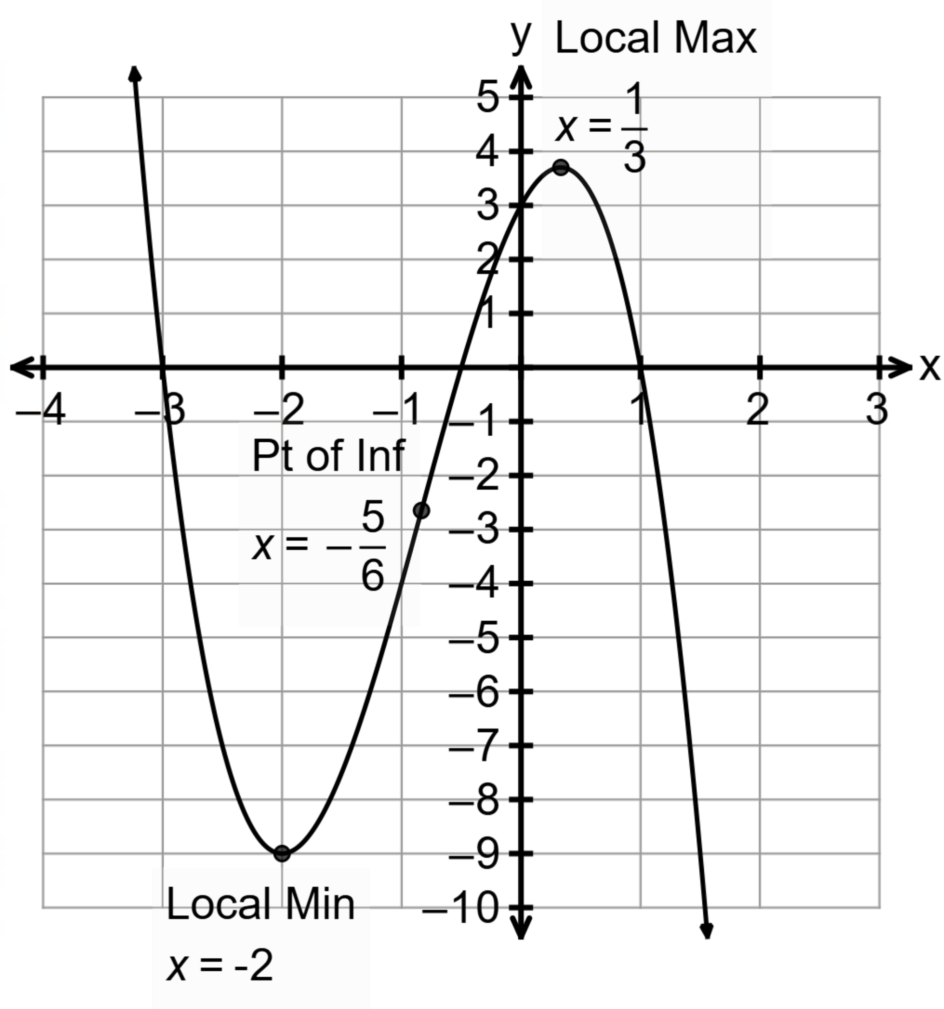
Consider our old friend the parabola

$f(x) = ax^2 + bx + c$

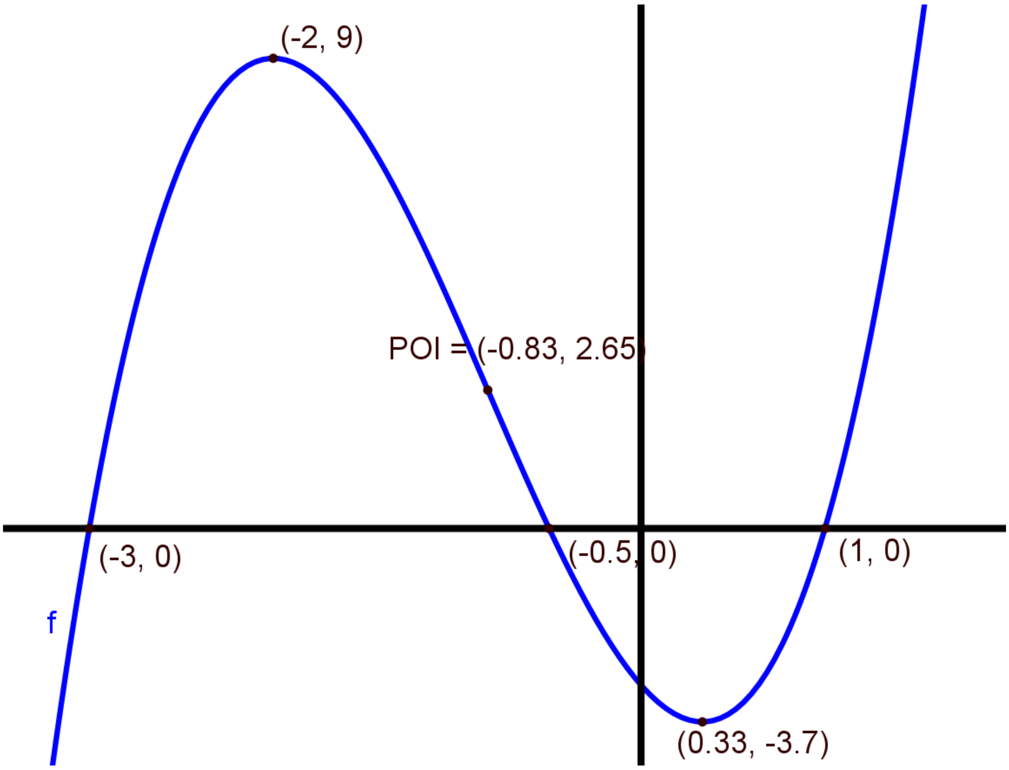
Points of inflection	Description	Gradient or "rate of change"	Gradient of the gradient or "rate of change of the rate of change"	How the graph Looks	What we call this	Stationary Points
	Symbol	$\frac{dy}{dx}$ or $f'(x)$	$\frac{d^2y}{dx^2}$ or $f''(x)$			
	function is stationary, gradient is increasing	$= 0$	> 0		A minimum turning point	Stationary Points
	function is stationary, gradient is decreasing	$= 0$	< 0		A maximum turning point	Stationary Points
Points of inflection	*function is stationary, gradient is stationary i.e. neither increasing nor decreasing but changing from decreasing to increasing(left picture) or increasing to decreasing (right picture)*	$= 0$	$= 0$	 or 	A horizontal point of inflection	Stationary Points
Points of inflection	*function is increasing, gradient is stationary i.e. neither increasing nor decreasing but changing from increasing to decreasing*	> 0	$= 0$		A point of inflection	
Points of inflection	*function is decreasing, gradient is stationary i.e. neither increasing nor decreasing but changing from decreasing to increasing*	< 0	$= 0$		A point of inflection	

So, given a point on a curve we can tell immediately, the signs of the function and both the first and second derivatives. Let's consider an example:

Give the signs (positive, negative or zero) of $f(x)$ and $f'(x)$ and $f''(x)$ at each of the given points.



x	-3	-2	-1	$-\frac{5}{6}$	0	$\frac{1}{3}$	1	$\frac{7}{6}$
$f(x)$								
$f'(x)$								
$f''(x)$								



Solve for x if:

$x.f'(x) > 0$

$\frac{f(x)}{f'(x)} \leq 0$

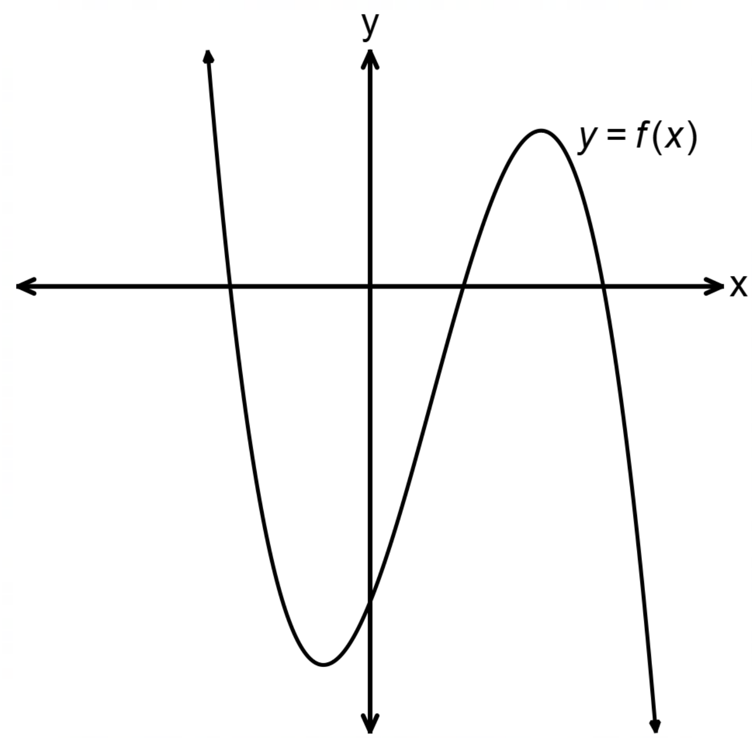
$f'(x)f''(x) < 0$

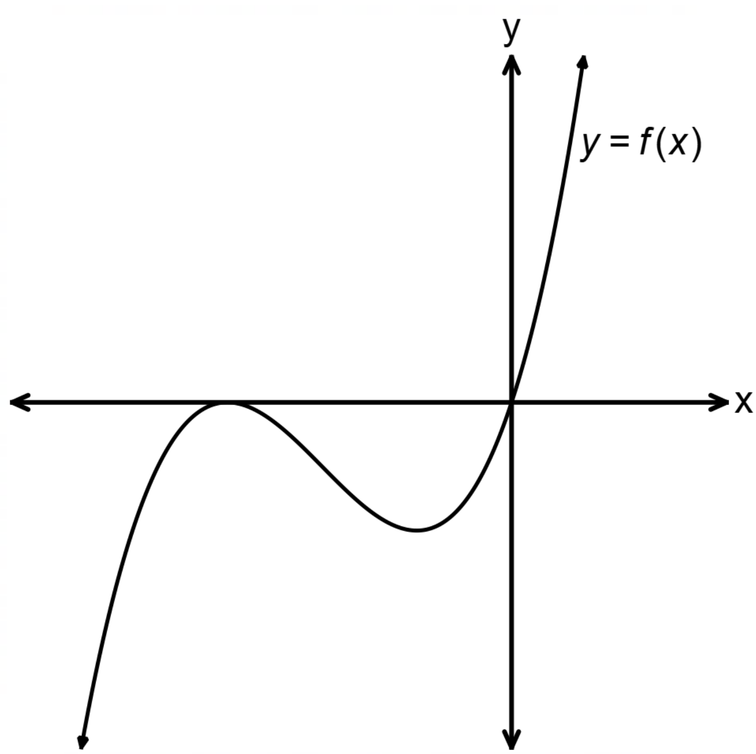
Find k if:

$f(x) + k = 0$ has only one root

$f(x + k) = 0$ has all three roots of the same sign

The other thing we can do is draw the graph of the derivative and second derivative of a function on the same set of axes as the function itself. Let's try it!

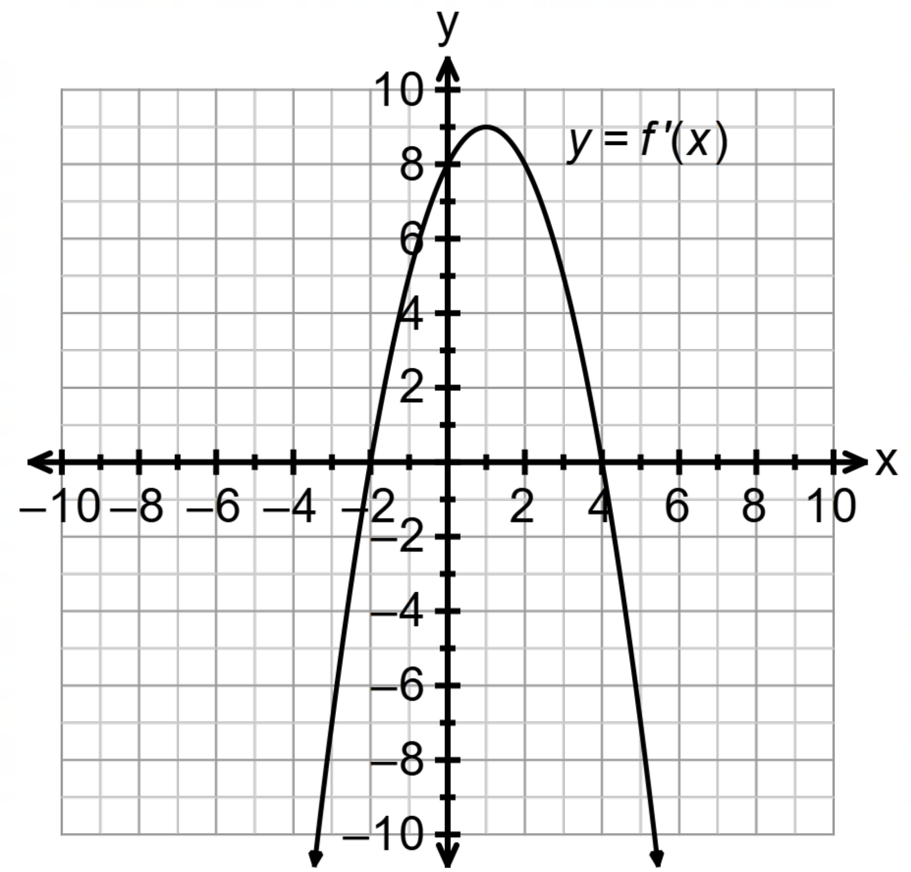




Sometimes we can be given the graph of a function and asked questions about the functions itself.

Try these.

1. The graph of $y = f'(x)$ is given.



a. For which value(s) of x is $f(x)$ increasing?

b. For which value(s) of x is $f(x)$ concave down?

c. Give the x-coordinates of any turning points of f.

d. Give the x-coordinate of the point of inflection of f.

MORE RATES OF CHANGE

Displacement, Velocity and Acceleration

- Differentiate displacement with respect to time $\left(\frac{ds}{dt}\right)$ to get velocity
- To get maximum displacement make velocity equal to zero.
- Differentiate velocity with respect to time $\left(\frac{dv}{dt} \quad or \quad \frac{d^2s}{dt^2}\right)$ to get acceleration

FOR THE SCIENTISTS:

$s = ut + \frac{1}{2}at^2$

$\frac{ds}{dt} =$

$\frac{dv}{dt} = \frac{d^2s}{dt^2} =$

1. A stone is thrown vertically upwards. Its height (h in meters) at time t (in seconds) is given by $h = 30t - 5t^2$. Determine its:

   a. Average velocity in the first second

   b. Instantaneous velocity after 1.5 seconds

   c. Acceleration

2. A particle covers a distance s (meters) in *t* (seconds) according to the formula:

$$s = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t$$

a. Give an expression for the velocity v in terms of t.

b. Determine the velocity after 2 seconds.

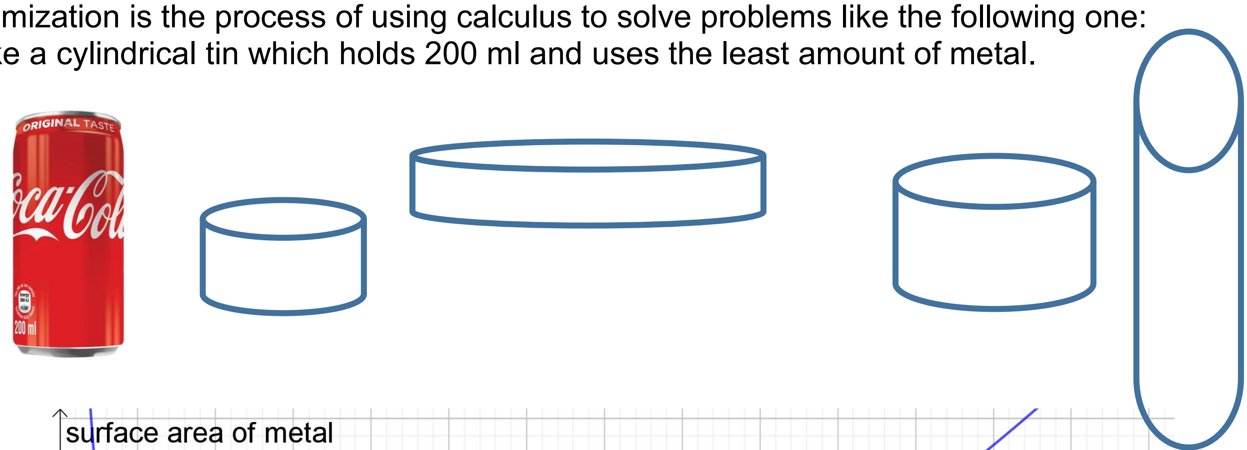
c. After how many seconds will the particle be stationary?

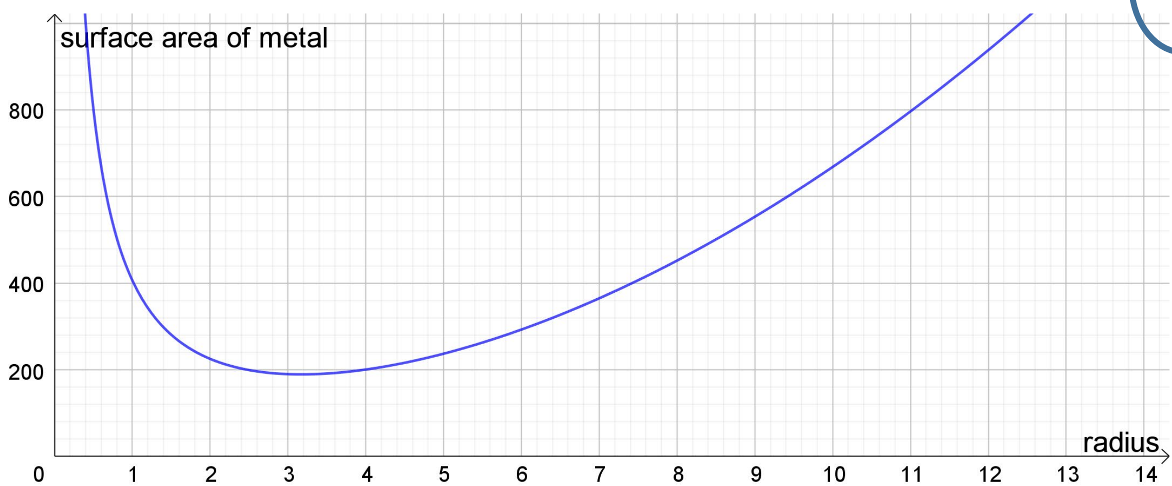
d. Express the acceleration a in terms of t.

e. Determine the acceleration after 3 seconds.

OPTIMIZATION

Optimization is the process of using calculus to solve problems like the following one:
Make a cylindrical tin which holds 200 ml and uses the least amount of metal.





To find the **optimum** value of something means to find its best value. Depending on the context this may be the biggest value (e.g., profits) or the smallest value (e.g., costs)

Calculus is centrally important because it helps us find the optimum value (typically where the derivative is zero).

We solve an optimization problem as follows:

- ✓ Make sure that you understand the problem. A sketch is often useful.
- ✓ Introduce a variable (suppose it is *x*) and get the other variables in terms of it.
- ✓ Determine an expression for the commodity that we are attempting to optimize. Suppose it is *M.*
- ✓ Differentiate the expression $\frac{dM}{dx}$ with respect to the relevant variable.
- ✓ Equate the derivative to zero and solve for the variable.
- ✓ Determine whether we have found a maximum or a minimum. We use the second derivative for this!
- ✓ Calculate the actual optimum if asked for. This is done by substituting back into the original equation.

The process is best explained with an example:

8. Consider the right-angled triangle CDE which has D at the origin and a vertex C on the line joining A to B. As shown. What is the maximum possible area of $\triangle DCE$.

Hint: let the point E have coordinates $(e;0)$