Resource for Day 4 - Calculus – Tuesday 24 October

1. Given $f(x)=(2x+3)^2$, determine $f'(x)$ by first principles. (5)

$f(x)=4x^2+12x+9$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{4(x+h)^2+12(x+h)+9-(4x^2+12x+9)}{h}$$

$$=\lim_{h\to 0}\frac{4(x^2+2xh+h^2)+12x+12h+9-4x^2-12x-9}{h}$$

$$=\lim_{h\to 0}\frac{4x^2+8xh+4h^2+12x+12h+9-4x^2-12x-9}{h}$$

$$=\lim_{h\to 0}\frac{8xh+4h^2+12h}{h}$$

$$=\lim_{h\to 0}\frac{h(8x+4h+12)}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}^1(8x+4h+12)}{\cancel{h}^1}$$

$$=\lim_{h\to 0}(8x+4h+12)$$

$$=8x+12$$

2. Given $f(x)=\frac{1}{x}$, determine $f'(x)$ by first principles. (6)

$$f(x)=\frac{1}{x}$$

$$f'(x)=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{h}$$

$$=\lim_{h\to 0}\frac{-\frac{h}{x(x+h)}}{h}$$

$$=\lim_{h\to 0}\left(-\frac{1}{x(x+h)}\right)$$

$$=-\frac{1}{x^2}$$

3. Determine the following derivatives, giving your answers with positive exponents where necessary. Remember to pay careful attention to notation!

a. $y=\frac{x^3+5x^2+15x+18}{2x+4}$ (4)

$$y=\frac{(x+2)(x^2+3x+9)}{2(x+2)}$$

$$y=\frac{\cancel{(x+2)}^{1}(x^2+3x+9)}{2\cancel{(x+2)}^{1}}$$

$$y=\frac{1}{2}x^2+\frac{3}{2}x+\frac{9}{2}$$

$$\frac{dy}{dx}=x+\frac{3}{2}$$

b. $D_x\left[\sqrt[5]{x^7}+\frac{4}{x^3}+\sqrt{2}\right]$ (4)

$$=D_x\left[x^{\frac{7}{5}}+4x^{-3}+\sqrt{2}\right]$$

$$=\frac{7}{5}x^{\frac{2}{5}}-12x^{-4}$$

$$=\frac{7.\sqrt[5]{x^2}}{5}-\frac{12}{x^4}$$

changing the surd back is nice but not necessary

c. $\frac{d}{dx}\left(\frac{4x^9+5x}{2x^7}\right)$ (3)

$$=\frac{d}{dx}\left(\frac{4x^9}{2x^7}+\frac{5x}{2x^7}\right)$$

$$=\frac{d}{dx}\left(2x^2+\frac{5}{2}x^{-6}\right)$$

$$=4x-15x^{-7}$$

$$=4x-\frac{15}{x^7}$$

4. Determine whether $f(x) = x - \sqrt{x}$ is increasing or decreasing or stationary when $x = 2$? Justify your answer. (3)

$f(x) = x - x^{\frac{1}{2}}$

$f'(x) = 1 - \frac{1}{2}x^{-\frac{1}{2}}$

$f'(x) = 1 - \frac{1}{2\sqrt{x}}$

$f'(2) = 1 - \frac{1}{2\sqrt{2}}$

$f'(2) = 0{,}64644661$

increasing since $f'(2) > 0$

5. Determine for which value(s) of x the function $y = 2x^3 + 5x^2 - 4x - 3$ is increasing? (5)

$y = 2x^3 + 5x^2 - 4x - 3$

increasing means $\frac{dy}{dx} > 0$

$\frac{dy}{dx} = 6x^2 + 10x - 4 > 0$

$3x^2 + 5x - 2 > 0$

$(3x - 1)(x + 2) > 0$

$x < -2$ or $x > \frac{1}{3}$ (concave up parabolas are positive outside their roots!)

6. Determine for which value(s) of x the function $f(x) = \frac{x^4}{12} + \frac{x^3}{6} - x^2 + 8x + 4$ is concave down. (5)

concave down means $f''(x) < 0$

$f(x) = \frac{x^4}{12} + \frac{x^3}{6} - x^2 + 8x + 4$

$f'(x) = \frac{4x^3}{12} + \frac{3x^2}{6} - 2x + 8$

$f''(x) = x^2 + x - 2$

$\therefore (x + 2)(x - 1) < 0$

$\therefore -2 < x < 1$ (concave up parabolas are negative between their roots)

7. Find the equation of the tangent to the function $y = \frac{6}{x} - 2$ at its x-intercept. (4)

x-intercept is 3

$y = 6x^{-1} - 2$

$\frac{dy}{dx} = -6x^{-2} = -\frac{6}{x^2}$

when $x = 3$ $\frac{dy}{dx} = -\frac{6}{9} = -\frac{2}{3}$

now, point of contact is (3;0)

$y - 0 = -\frac{2}{3}(x - 3)$

$y = -\frac{2}{3}x + 2$

8. The function $f(x) = x^3 - x^2 - x - 1$ has two tangents which are parallel to the line $2y - 8x = 16$. Find both of their equations. (6)

$2y - 8x = 16$

is $2y = 8x + 16$

is $y = 4x + 8$

so, our tangents must each have a gradient of 4

$f'(x) = 3x^2 - 2x - 1 = 4$

$3x^2 - 2x - 5 = 0$

$(3x - 5)(x + 1) = 0$

so, one point of contact is $\frac{5}{3}$ and the other is -1

$f\left(\frac{5}{3}\right) = \left(\frac{5}{3}\right)^3 - \left(\frac{5}{3}\right)^2 - \left(\frac{5}{3}\right) - 1 = -\frac{22}{27}$

$f(-1) = (-1)^3 - (-1)^2 - (-1) - 1 = -2$

① $y + \frac{22}{27} = 4\left(x - \frac{5}{3}\right)$ or $y = 4x - \frac{202}{27}$

② $y + 2 = 4(x + 1)$ or $y = 4x + 2$

9. The function $f(x) = x^3 + bx^2 + cx + 5$ has a local minimum at $(2;-3)$. Find b and c. (6)

$f(x) = x^3 + bx^2 + cx + 5$ so $f'(x) = 3x^2 + 2bx + c$

we know $f(2) = -3$ ① and $f'(2) = 0$ ②

from ① $8 + 4b + 2c + 5 = -3$
$4b + 2c = -16$ ③

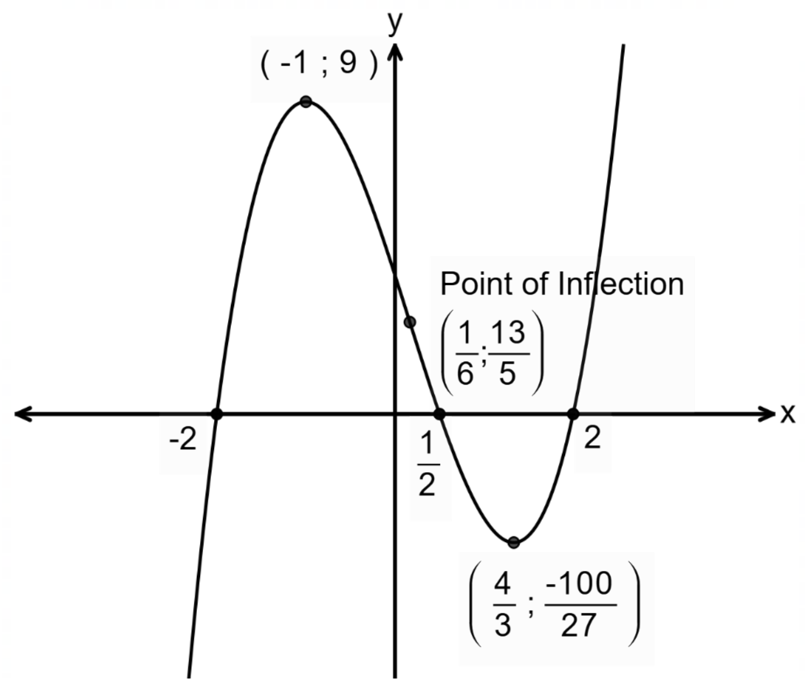
from ② $12 + 4b + c = 0$
$4b + c = -12$ ④

③ – ④ gives:
$c = -4$ and substituting this gives:
$b = -2$

10. Determine the **average gradient** of the function $y = 2^x - 8$ between its intercepts with the axes. (4)

y-intercept (0;-7)
x-intercept (3;0)
average $gradient = \frac{7}{3}$

11. Consider the graph of the function $f(x)$ shown below and use it to solve the equations and inequalities which follow:



Solve for *x*:

a. $f'(x)<0$ (2)

decreasing when $-1<x<\frac{4}{3}$

b. $f''(x)=0$ (1)

point of inflection so $x=\frac{1}{6}$

c. $xf(x)\leq 0$ (4)

we need a negative product which will result from one positive factor and one negative

① $x\leq 0$ (to left of y-axis) and $f(x)\geq 0$ (above x-axis)

$-2\leq x\leq 0$

OR

② $x\geq 0$ (to right of y-axis) and $f(x)\leq 0$ (below x-axis)

$\frac{1}{2}\leq x\leq 2$

d. $f'(x)f''(x)>0$

we need a positive product which will result from both factors having the same sign!

① $f'(x)>0$ (increasing) and $f''(x)>0$ (concave up)

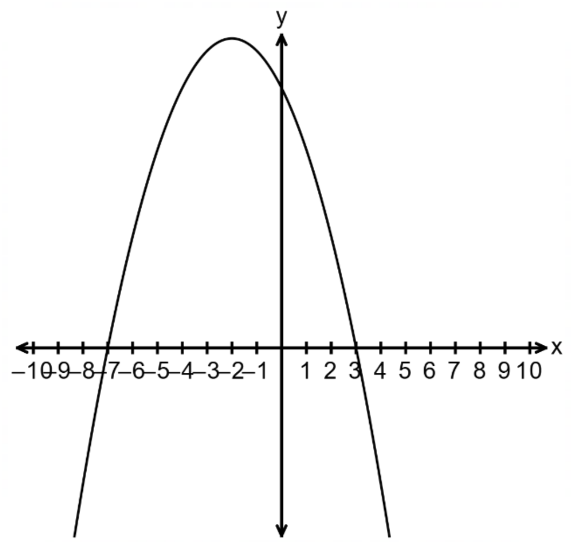
$x>2$

② $f'(x)<0$ (decreasing) and $f''(x)<0$ (concave down)

$-1<x<\frac{1}{6}$

(4)

12. The graph of $f'(x)$ is drawn.



a. Determine the x-coordinates of the stationary points of $f(x)$ and justify the nature of the stationary points. (4)

the stationary points are at x = -7 and at x = -3

there is a local minimum where x = -7
since the gradient goes from negative through zero to positive

there is a local minimum where x = 3
since the gradient goes from positive through zero to negative

b. For which value(s) of x is $f(x)$ concave down? (1)

$f(x)$ will be concave down where $f''(x)<0$
this is where the gradient of $f'(x)<0$
this is $x>$-2 (found using the symmetry of the parabola!)

13. Sibonelo takes a piece of rope 100 cm long. He cuts it into two pieces and makes a square with the one piece and circle with the other. If the length of the side of the square is x then:

a. Determine an expression for the total area enclosed by the two shapes. (3)

area of $square = x^2$

rope left for circle = $100 - 4x$

but this is the circumference

$c = 2\pi r$

so $r = \frac{c}{2\pi} = \frac{100 - 4x}{2\pi}$

$A = \pi r^2$

$A = \pi \left(\frac{100 - 4x}{2\pi} \right)^2$

$A = \pi \left(\frac{10\,000 - 800x + 16x^2}{4\pi^2} \right)$

$A = \frac{10\,000 - 800x + 16x^2}{4\pi}$

$A = \frac{10\,000 - 800x + 16x^2}{4\pi}$

$A = \frac{2\,500}{\pi} - \frac{200}{\pi}x + \frac{4}{\pi}x$

b. Determine x if Sibonelo wants the total area enclosed by the two shapes to be as small as possible. (4)

$A = x^2 + \frac{2\,500}{\pi} - \frac{200}{\pi}x + \frac{4}{\pi}x^2$

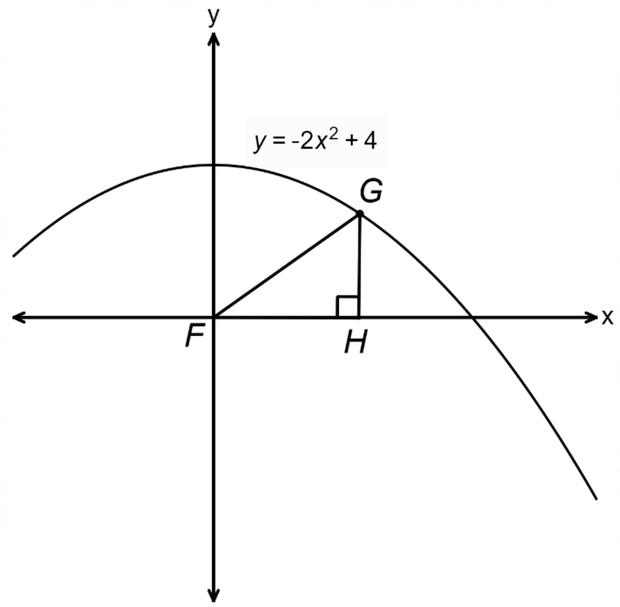
$\frac{dA}{dx} = 2x - \frac{200}{\pi} + \frac{8}{\pi}x = 0$

$x\left(2 + \frac{8}{\pi} \right) = \frac{200}{\pi}$

$x = \frac{200}{\pi} \div \left(2 + \frac{8}{\pi} \right)$

$x = 14$ cm

14. A right angled ΔFGH with $\hat{H} = 90°$ and F at the origin has G on the curve $y = -2x^2 + 4$ as shown. The x-coordinate of G *is positive.*



Determine the x-coordinate of G if ΔFGH is to have maximum area. (6)

suppose $G(x;-2x^2+4)$ $NOTE$: every point which lies ***ON*** *f has this form!*

then base of $\Delta FGH = x$ and height of $\Delta FGH = -2x^2 + 4$

Area $\Delta FGH = \frac{1}{2}x(-2x^2+4)$

$A = -x^3 + 2x$

$\frac{dA}{dx} = -3x^2 + 2 = 0$

$3x^2 = 2$

$x^2 = \frac{2}{3}$

$x = \sqrt{\frac{2}{3}}$

$x \approx 0{,}8165$ and, just for fun, max area is $-\left(\frac{\sqrt{2}}{3}\right)^3 + 2\left(\frac{\sqrt{2}}{3}\right) = 1{,}0887\ u^2$