Resource for Day 5 – Sequences and Series – Wednesday 25 October

1. Give the next three terms in each of the following sequences:

   a. $14a + 3$; $11a + 8$; $8a + 13$; $5a + 18$; .... (2)

   $2a + 23$ and $-a + 28$

   b. $2\,000y^2$; $-1\,000y^3$; $500y^4$; ..... (2)

   $-250y^5$ and $125y^6$

   c. 2 ; 5 ; 10 ; 17 ; .... (2)

   26 and 37

2. Is -21 a term in the sequence 19 ; $17\frac{1}{2}$; 16 ; $14\frac{1}{2}$; ....? (3)

$$T_n = a + (n-1)d$$
$$-21 = 19 + (n-1)\left(-\frac{3}{2}\right)$$
$$-40 = (n-1)\left(-\frac{3}{2}\right)$$
$$\frac{80}{3} = n - 1$$
$$n = \frac{80}{3} + 1$$
$$n = 27{,}7$$

no, since $n \notin \mathbb{N}$

3. A geometric sequence has $T_8 = 2\,025$ and $T_{12} = 400$. Find two possible value(s) for r (4)

$$T_n = ar^{n-1}$$
$$400 = ar^{11} \quad ①$$
$$2\,025 = ar^7 \quad ②$$

dividing ① by ② gives:

$$\frac{400}{2\,025} = r^4$$
$$r = \pm\left(\frac{400}{2\,025}\right)^{\frac{1}{4}}$$
$$r = \pm\frac{2}{3}$$

4. An arithmetic sequence has $T_9 = 28$ and $S_{20} = 650$. Find the first three terms. (6)

since $T_n = a + (n-1)d$ we have $28 = a + 8d$ ①

since $S_n = \frac{n}{2}[2a + (n-1)d$ we have $650 = \frac{20}{2}[2a + 19d]$

from ① we have $a = 28 - 8d$

substituting this into ② gives:

$650 = 10[2(28 - 8d) + 19d]$

$65 = 56 - 16d + 19d$

$9 = 3d$

$d = 3$ and $a = 4$

so, first three terms are 4 ; 7 and 10

5. Determine g if $g + 3$; $4g + 2$; $5g + 10$ are consecutive terms of a geometric sequence and $g \in \mathbb{Z}$. (5)

$\frac{4g+2}{g+3} = \frac{5g+10}{4g+2}$

$(4g+2)^2 = (5g+10)(g+3)$

$16g^2 + 16g + 4 = 5g^2 + 25g + 30$

$11g^2 - 9g - 26 = 0$

$(11g + 13)(g - 2) = 0$

$g = -\frac{13}{11}$ or $g = 2$

but $g \in \mathbb{Z}$ so $g = 2$ only

6. A SIMI sequence is one in which the terms in odd-numbered positions form an **arithmetic sequence** while the terms in even-numbered positions form a **geometric sequence**.

Consider this SIMI sequence: $2\ ;\ \frac{1}{4}\ ;\ 5\ ;\ \frac{1}{2};\ 8\ ;\ 1\ ;\ 11\ ;\ 2\ ;\ 14\ ;\ ....$

a. write down the next two terms of the above SIMI sequence. (1)

4 and 17

b. find T_{21} of the above SIMI sequence. (3)

T_{21} of the SIMI sequence will be T_{11} of the arithmetic sequence

$T_{11} = 2 + 10(3) = 32$

c. Find S_{19} of the above SIMI sequence. (6)

this will be made up of:

9 terms of the geometric sequence and 10 of the arithmetic one

$$S_{19} = \frac{10}{2}[2(2) + 9(3)] + \left(\frac{\frac{1}{4}(2^9 - 1)}{2 - 1} \right)$$

$S_{19} = 282,75$

7. Determine p if $\sum_{i=7}^{30} (pi - 1) = 1\ 308$ (5)

$$\sum_{i=7}^{30} (pi - 1) = (7p - 1) + (8p - 1) + (9p - 1) + ...to\ 24 \text{ terms} = 1\ 308$$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$1\ 308 = \frac{24}{2}[2(7p - 1) + 23p]$

$1\ 308 = 12[14p - 2 + 23p]$

$109 = -2 + 37p$

$111 = 37p$

$p = 3$

8. A plant grows 90 cm in the first year. Each year thereafter it grows by 15% of the amount it grew the year before. What is the greatest height it can ever (never 😉) reach? (5)

total height of plant will be the sum of the amounts it grew each year:

$90 + (90 \times 0{,}15) + (90 \times 0{,}15^2) + (90 \times 0{,}15^3) + ...$

this is a convergent geometric series since $r = 0{,}15$ and $-1 < r < 1$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{90}{1-0{,}15}$$

$$S_\infty = 105{,}88\text{cm}$$

9. How many hexagons in figure n? (5)

fig. 1

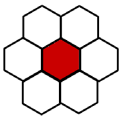


fig. 2

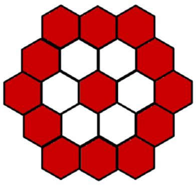


fig. 3

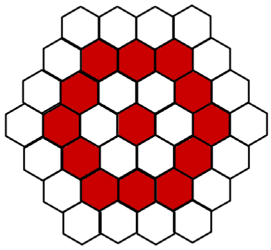


fig. 4

the sequence is: 1 ; 7 ; 19 ; 37 ; ....

first differences are 6 ; 12 ; 18 ; ....

second differences are 6 ; 6 ; 6 ; ....

so $a = \frac{6}{2} = 3$

now $3a + b = 6$

so $3(3) + b = 6$

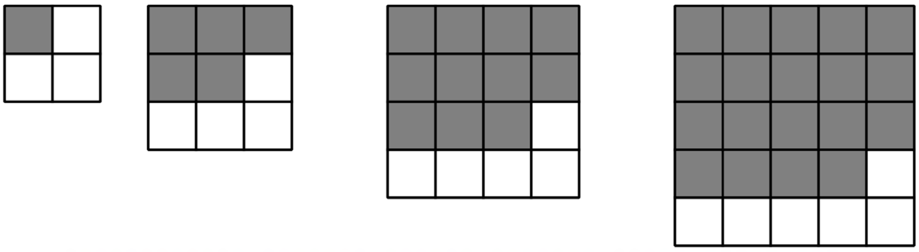
so $b = -3$

now $a + b + c = 1$

so $c = 1$

$T_n = 3n^2 - 3n + 1$

10. Which figure will have 599 shaded squares? (6)



This one can be done by inspection if one looks at the structure!
For example, in figure 2, we have a total of 3^2 (one more) squares
from this we must subtract 3 and 1

So, $T_n = (n+1)^2 - (n+1) - 1$

$T_n = n^2 + n - 1$

alternatively....

1 ; 5 ; 11 ; 19 ; ....

first differences 4 ; 6 ; 8 ; ....

second differences 2 ; 2 ; 2 ; ....

$a = 1$

$3(1) + b = 4$

so $b = 1$

$1 + 1 + c = 1$

so $c = -1$

$T_n = n^2 + n - 1 = 599$

$n^2 + n - 600 = 0$

$(n+25)(n-24) = 0$

$n = 24$

11. The **first differences** of a quadratic sequence are given by $T_n = 1 - 2n$. If the third term of the quadratic sequence is 5 then find a formula for T_n, its n^{th} term. (6)

first differences are -1 ; -3 ; -5 ; .....

second differences are -2 ; -2 ; -2 ; ....

$a = -1$

$3(-1) + b = -1$

so $b = 2$

now $T_n = -n^2 + 2n + c$

but we know that $T_3 = 5$

$5 = -9 + 6 + c$

so, $c = 8$

$T_n = -n^2 + 2n + 8$

12. A quadratic sequence has a constant second difference of 2 and $T_5 = 39$ and $T_{10} = 124$. Find a formula for T_n, its n^{th} term. (6)

since constant second difference is 2, $a = 1$

$T_n = n^2 + bn + c$

Since $T_5 = 39$ we know $39 = 25 + 5b + c$ ①

Since $T_{10} = 124$ we know $124 = 100 + 10b + c$ ②

② – ① gives:

$85 = 75 + 5b$

$10 = 5b$

so $b = 2$

and $c = 4$

$\therefore T_n = n^2 + 2n + 4$

13. For which values of x will the series $\sum_{i=1}^{\infty} (-2x+3)^i$ converge? (4)

$$\sum_{i=1}^{\infty} (-2x+3)^i = (-2x+3) + (-2x+3)^2 + (-2x+3)^3 + \ldots$$

so, $r = -2x + 3$

for convergence $-1 < r < 1$ so

$-1 < -2x + 3 < 1$

$-4 < -2x < -2$

$2 > x > 1$

$1 < x < 2$

14. The sum of 20 terms of an arithmetic series is 595 and $T_{n+2} - T_n = 5$ for all $n \in \mathbb{N}$. Find the first three terms. (5)

$T_{n+2} - T_n = 5$ for all $n \in \mathbb{N}$

means, for example, that $T_3 - T_1 = 5$

this means that the gap between 2 terms is 5

so $d = \frac{5}{2}$

now $S_n = \frac{n}{2}[2a + (n-1)d]$

so, $595 = \frac{20}{2}\left[2a + 19\left(\frac{5}{2}\right)\right]$

$59{,}5 = 2a + 47{,}5$

$12 = 2a$

$a = 6$

first three terms are 6 ; $8\frac{1}{2}$; 11